نموذج أجابة

رياضيات هندسية (٢-ب)

Engineering Mathematic (2)B

الترم الأول

قسم مدني + ميكانيكا + ك

2015- 2016

تاريخ الأمتحان : 10/1/2016

د/ منال السيد على

10/1/2016

Engineering Mathematics (2)

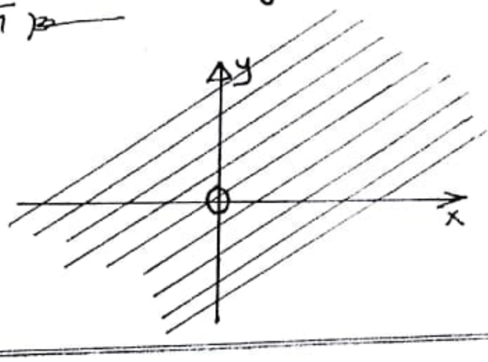
Question [1] :-

(a) find the domain for a function $f(x,y) = (x^2-y^2)^{-3/2}$

— solution —

$$f(x,y) = \frac{1}{(x^2-y^2)\sqrt{x^2-y^2}}$$

$$D_f = \{(x,y) : x^2-y^2 > 0\}$$



(b) Discuss the continuity for a function $f(x,y) = \begin{cases} \frac{\sin(x+y)}{x+y} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

— solution —

① $f(0,0) = 1$

② $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = \frac{0}{0}$ كمية غير معينة

* $y = mx$

$$\lim_{x\to 0} \frac{\sin(x+mx)}{x+mx} = \lim_{x\to 0} \frac{(1+m)\cos(x+mx)}{1+m} = 1 = L_1$$

$$* \lim_{x\to 0}\left[\lim_{y\to 0} \frac{\sin(x+y)}{x+y}\right] = \lim_{x\to 0} \frac{\sin x}{x} = 1 = L_2$$

$$* \lim_{y\to 0}\left[\lim_{x\to 0} \frac{\sin(x+y)}{x+y}\right] = \lim_{y\to 0} \frac{\sin y}{y} = 1 = L_3$$

* $y = x^2$

$$\lim_{x\to 0} \frac{\sin(x+x^2)}{x+x^2} = \lim_{x\to 0} \frac{(1+2x)\cos(x+x^2)}{(1+2x)} = 1 = L_4$$

$\because L_1 = L_2 = L_3 = L_4$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = 1$$

$$\because f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$$

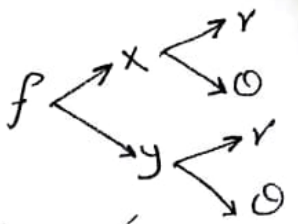
$\therefore f(x,y)$ is continous at $(0,0)$

(c) If $f = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

show that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$

— Solution —

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta$$

$$\left(\frac{\partial f}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + \cancel{2\,\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\sin\theta\cos\theta} \rightarrow (1)$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial f}{\partial \theta}\right)^2 = r^2\sin^2\theta\left(\frac{\partial f}{\partial x}\right)^2 + r^2\cos^2\theta\left(\frac{\partial f}{\partial y}\right)^2 - 2r^2\sin\theta\cos\theta\,\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}$$

$$\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \sin^2\theta\left(\frac{\partial f}{\partial x}\right)^2 + \cos^2\theta\left(\frac{\partial f}{\partial y}\right)^2 - \cancel{2\sin\theta\cos\theta\,\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}} \rightarrow (2)$$

(1) + (2)

$$\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\underbrace{\left[\sin^2\theta + \cos^2\theta\right]}_{1} + \left(\frac{\partial f}{\partial y}\right)^2\underbrace{\left[\sin^2\theta + \cos^2\theta\right]}_{1}$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \quad \#$$

d) prove that $I = \int^2 \int_0^{2-y} \frac{dx\,dy}{(x+y)^2} = \ln(2) - \frac{1}{2}$

— Solution —

$$I = \int_1^2 \int_0^{2-y} \frac{1}{(x+y)^2}\,dx\,dy = \int_1^2 \frac{-1}{(x+y)^{+1}}\Big|_0^{2-y} dy$$

$$= \int_1^2 \frac{-1}{2} + \frac{1}{y}\,dy = -\frac{1}{2}y + \ln y\Big|_1^2 = -1 + \ln 2 + \frac{1}{2} - \cancel{\ln 1}_{0}$$

$$= \ln 2 - \frac{1}{2} \quad \#$$

Question [2]

a) Find Maclurin expansion for function $f(x,y) = e^{ax+by}$, a, b constants.

— solution —

$f(x,y) = e^{ax+by} \longrightarrow f(0,0) = 1$

$f_x(x,y) = a\, e^{ax+by} \longrightarrow f_x(0,0) = a$

$f_y(x,y) = b\, e^{ax+by} \longrightarrow f_y(0,0) = b$

$f_{xx}(x,y) = a^2 e^{ax+by} \longrightarrow f_{xx}(0,0) = a^2$

$f_{xy}(x,y) = ab\, e^{ax+by} \longrightarrow f_{xy}(0,0) = ab$

$f_{yy}(x,y) = b^2 e^{ax+by} \longrightarrow f_{yy}(0,0) = b^2$

$$f(x,y) = f(0,0) + x\, f_x(0,0) + y\, f_y(0,0) + \frac{1}{2}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

$$e^{ax+by} = 1 + ax + by + \frac{1}{2}\left[a^2x^2 + 2ab\,xy + b^2y^2\right] + \cdots$$

b) prove that, the expression $(3x^2y - 2y^2)dx + (x^3 - 4xy + 6y^2)dy$ represent total differential for $f(x,y)$ and find the function.

— solution —

$M = (3x^2y - 2y^2)$ $\qquad$ $N = x^3 - 4xy + 6y^2$

$\frac{\partial M}{\partial y} = 3x^2 - 4y$ $\qquad$ $\frac{\partial N}{\partial x} = 3x^2 - 4y$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ Exact (total differential)

$df(x,y) = (3x^2y - 2y^2)\,dx + (x^3 - 4xy + \underline{6y^2})\underline{dy}$

$f(x,y) = \int (3x^2y - 2y^2)dx + (x^3 - 4xy + \underline{6y^2})\underline{dy}$

$= \underline{x^3y} - 2xy^2 + \underline{x^3y} - 2xy^2 + 2y^3$

$f(x,y) = x^3y - 2xy^2 + \underline{2y^3}$

c) Find the maximum and minimum value (if exist) for function $f(x,y) = \frac{x^2}{2a} + \frac{y^2}{2b}$ $a \& b > 0$

— Solution —

① $f_x = 0$

$\frac{2x}{2a} = 0 \longrightarrow x = 0$

$f_y = 0$

$\frac{2y}{2b} = 0 \longrightarrow y = 0$

critical point is $(0,0)$

② $f_{xx} = \frac{1}{a}$, $f_{xx}\Big|_{(0,0)} = \frac{1}{a}$

$f_{xy} = 0$, $f_{xy}\Big|_{(0,0)} = 0$

$f_{yy} = \frac{1}{b}$, $f_{yy}\Big|_{(0,0)} = \frac{1}{b}$

③ $\Delta = f_{xx} * f_{yy} - (f_{xy})^2$

$= \frac{1}{a}\,\frac{1}{b} - 0 = \frac{1}{ab} > 0$

if $a \& b$ are positive constant or a, b are negative constant

$\therefore (0,0)$ is Minimum point

$\therefore$ Min value $= \frac{(0)^2}{2a} + \frac{(0)^2}{2b} = 0$

d) Find $V = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz\,dy\,dx$

— Solution —

$V = \int_0^a \int_0^{a-x} z\Big|_0^{a-x-y} dy\,dx = \int_0^a \int_0^{a-x} a-x-y \; dy\,dx$

$= \int_0^a \left(a - x)y - \frac{y^2}{2}\right|_0^{a-x} dx = \int_0^a (a-x)^2 - \frac{(a-x)^2}{2}\, dx$

$= \frac{-1}{2}\int_0^a -(a-x)^2\, dx = -\frac{1}{2}\,\frac{(a-x)^3}{3}\Big|_0^a = -\frac{1}{2}\left[-\frac{a^3}{3}\right] = \frac{a^3}{6}$

$\boxed{Q3}$ a) ~~ist~~ Find solution of

i) $y' = \frac{1+2xy}{x^2} \rightarrow x^2 y' = 1 + 2xy \rightarrow y' = \frac{1}{x^2} + \frac{2y}{x}$

$y' - \frac{2}{x} y = \frac{1}{x^2}$

$\mu = e^{\int p(x)\,dx} = e^{-2\int \frac{1}{x} dx} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

$y = x^2\left[\int \frac{1}{x^2} \cdot \frac{1}{x^2} dx + c\right] = x^2\left[\frac{x^{-3}}{-3} + c\right] = \frac{x^{-1}}{-3} + cx^2$

$y = \frac{-1}{3x} + cx^2$

ii) $y\,dx = (x + \sqrt{y^2 - x^2})\,dy$

$\frac{dy}{dx} = \frac{y}{x + \sqrt{y^2 - x^2}}$ Hom of dgree 1

$\frac{dy}{dx} = \frac{y/x}{1 + \sqrt{(y/x)^2 - 1}} \longrightarrow$ let $v = y/x$ $\quad \frac{dy}{dx} = v + v'x$

$\cancel{v} + v'x = \frac{v}{1 + \sqrt{v^2 - 1}} - \cancel{v} = \frac{-v\sqrt{v^2-1}}{1 + \sqrt{v^2 - 1}}$

$\int \frac{1 + \sqrt{v^2-1}}{-v\sqrt{v^2-1}} dv = \int \frac{dx}{x}$

$\int \frac{-1}{v\sqrt{v^2-1}} dv - \int \frac{\sqrt{v^2-1}}{v\sqrt{v^2-1}} dv = \ln|x| + c$

$-\sec^{-1} v - \ln|v| = \ln|x| + c$

$-\sec^{-1}(y/x) - \ln|y/x| = \ln|x| + c$

iii) $\cosh x \sin y\, dy = \sinh x \cos y\, dx$

$\underbrace{\sinh x \cos y}_{M}\, dx - \underbrace{\cosh x \sin y}_{N}\, dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\sinh x \sin y \Rightarrow$ exact equation

Let $u(x,y) = \int M(x,y)\,dx + f(y) = \int \sinh x \cos y \, dx + f(y)$

$= \cosh x \cos y + f(y) \longrightarrow *$

$\frac{\partial u}{\partial y} = -\cosh y \sin y + \frac{d f(y)}{dy} = N(x,y)$

$= -\cosh x \sin y$

$\therefore \frac{d f(y)}{dy} = 0 \longrightarrow \int d f(y) = \int 0 \, dy$.

$f(y) = c$

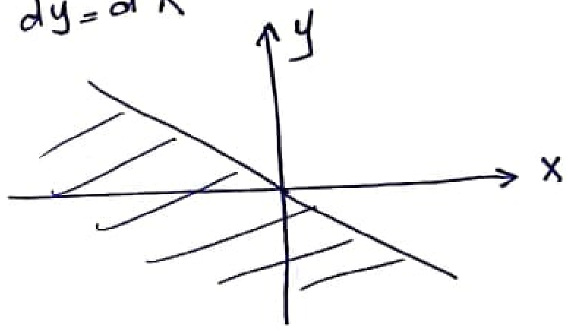
$\boxed{\cosh x \cos y + c = 0}$

b) Find family of curves (explain by graph)

$y = -x$

$y' = -1 \longrightarrow y' = 1 \longrightarrow \frac{dy}{dx} = 1 \quad dy = dx$

$y = x + c$



c) Find general solution

i) $(D^5 - 32)y = 0 \longrightarrow \alpha^5 - 32 = 0 \quad \alpha^5 = 32 \quad r = 32 \quad \theta = 0$

$\alpha^5 = re^{i\theta} = 32\, e^{i2\pi k} = 32\,[\cos 2\pi i + i \sin 2\pi k] \quad k = 0, 1, 2, 3, 4$

at $k=0 \longrightarrow \alpha_1 = 2e^0 = 2 \longrightarrow$ real

$k=1 \longrightarrow \alpha_2 = 2e^{i\frac{2\pi k}{5}} = 2[\cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}] = 0.618 + i\,1.9$

$k=2 \longrightarrow \alpha_3 = 2e^{i4\pi/5}$ ______ $= -1.6 + i\,1.175$

$k=3 \longrightarrow \alpha_4 = 2e^{i6\pi/5} =$ ______ $= -1.6 - i\,1.175$

$k=4 \longrightarrow \alpha_5 = 2e^{i8\pi/5} =$ ______ $= 0.618 - i\,1.9$

$\alpha_1 = 2 \longrightarrow$ real

$\alpha_{2,5} = 0.618 \pm i\,1.9$ Complex not repeated

$\alpha_{3,4} = -1.6 \pm 1.175$ " " "

$$y_{G.S} = C_1 e^{2x} + [A_1 \cos 1.9x + B_1 \sin 1.9x]\, e^{0.1618x} + e^{-1.6}[A_2 \cos 1.175x + B_2 \sin 1.175x]$$

ii) $(D+2)^3 (D-1)(D^2+1)y = 0$

$\alpha_1 = -2$ real repeated 3-times

$\alpha_2 = 1$ 〃 not 〃

$\alpha_{3,4} = \pm i$ complex not repeated

$$y_{G.S} = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 x^2 e^{-2x} + C_4 e^{x} + [A_1 \cos x + B_1 \sin x]$$

$\boxed{Q4}$ a) Deduce laplace transform $f(t) = \sin at$

$$F(s) = \int_0^\infty e^{-st} \sin at \, dt = I$$

$$I = \left[\frac{e^{-st}}{-s} \sin at - \frac{e^{-st}}{s^2} a \cos at \right]_0^\infty - \frac{a^2}{s^2} I$$

$$I + \frac{a^2}{s^2} I = \left[〃 \quad - 〃 \right]_0^\infty$$

$$I = \frac{s^2}{s^2+a^2}\left[\frac{a}{s^2}\right] = \frac{a}{s^2+a^2}$$

u	dv
$\sin at$	e^{-st}
$a \cos at$	$e^{-st}/-s$
$-a^2 \sin at$	e^{-st}/s^2
$+\int$	

b) i) $L[e^{3t} \sinh 5t] = \dfrac{5}{(s-3)^2 - 25}$

ii) $L[\cos^3 t] = L[\cos t \cdot \frac{1}{2}(1 + \cos 2t)] =$

$$= \frac{1}{2} L\left[\cos t + \frac{1}{2}(\cos 3t + \cos t)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} + \frac{1}{2}\left(\frac{s}{s^2+9} + \frac{s}{s^2+1}\right)\right]$$

iii) $L^{-1}\left[\frac{1}{4s^2-1}\right] = L^{-1}\left[\frac{1}{4}\left(\frac{1}{s^2-1/4}\right)\right] = L^{-1}\left[\frac{2}{4}\left(\frac{1/2}{s^2-(1/2)^2}\right)\right]$

$= \frac{1}{2} \sinh \frac{1}{2} t$

c) $\frac{dy}{dx} + ay = x \quad y(0) = 0$ using Laplace transform

$y' + ay = x$

$sY(s) - y(0) + aY(s) = \frac{1}{s^2}$

$Y(s)(s+a) = \frac{1}{s^2}$

$Y(s) = \frac{1}{s^2(s+a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+a}$

$A = \frac{-1}{a} \qquad B = \frac{1}{a} \qquad C = \frac{1}{a^2}$

$Y(s) = \frac{-1/a^2}{s} + \frac{1/a}{s^2} + \frac{1/a^2}{s+a}$

$y(x) = \frac{-1}{a^2} + \frac{1}{a} t + \frac{1}{a^2} e^{-at}$