نموذج إجابة

رياضيات هندسية (٣- أ)

Engineering Mathematic (3-A)

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1] a) $z_1 = 1+i$ $\qquad z_2 = 1+\sqrt{3}i$

$Re(z_1 z_2) = 1+\sqrt{3}i + i - \sqrt{3} = (1-\sqrt{3}) + i(1+\sqrt{3})$

$Im\left(\frac{z_1}{z_2}\right) = \frac{1+i}{1+\sqrt{3}i} * \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{1+\sqrt{3}}{1+3} + i\,\frac{1-\sqrt{3}}{4}$

b) Find real part u and imaginary part v

i) $f(z) = z\,\bar{e}^{z}$

$= (x+iy)\,\bar{e}^{x-iy} = (x+iy)\,\bar{e}^{x}(\cos(-y)) + i\sin((-y)))$

$= x\bar{e}^{x}\cos y + y\,\bar{e}^{x}\sin y) + i(y\,\bar{e}^{x}\cos y - x\,\bar{e}^{x}\sin y)$

ii) $f(z) = \ln z$

$= \ln re^{i\theta} = \ln r + \ln e^{i\theta} = \ln r + i\theta = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

d) i) $f(z) = \begin{cases} \frac{\bar{z}}{z} & z \neq 0 \\ 3 & z = 0 \end{cases}$ discuss the continuity

$\lim\limits_{z\to 0}\frac{\bar{z}}{z}$ is not exist , $f(0) = 3$

$\therefore \lim\limits_{z\to 0} f(z) \neq f(0) \Rightarrow \therefore f(z)$ not continuous at $z = 0$

c) i) $\lim\limits_{z\to 0}\frac{3x^2y}{x^2+y^2}$ put $y = mx$

$\lim\limits_{x\to 0}\frac{3mx^3}{x^2+m^2x^2} = \lim\limits_{x\to 0}\frac{3x}{(1+m^2)} = 0$ exist

ii) $\lim\limits_{z\to 0}\frac{\bar{z}}{z} \Rightarrow$

$\lim\limits_{x\to 0}\lim\limits_{y\to 0}\frac{x-iy}{x+iy} = \lim\limits_{x\to 0}\frac{x}{x} = 1 \longrightarrow (1)$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0}\frac{x-iy}{x+iy} = \lim\limits_{y\to 0}\frac{-iy}{iy} = -1 \longrightarrow (2)$

$(1) \neq (2)$ $\therefore$ limit not exist

Q2 a) prove $\frac{d}{dz}(\ln z) = \frac{1}{z}$

$\ln re^{i\theta} = \ln r + \ln e^{i\theta}$

$f(z) = \ln r + i\theta \Rightarrow u = \ln r \quad v = \theta$

$u_r = \frac{1}{r} \qquad v_r = 0$

$u_\theta = 0 \qquad v_\theta = 1 \Longrightarrow u_r = \frac{1}{r} v_\theta \quad , v_r = \frac{-1}{r} u_\theta$

$f'(z) = (u_r + i v_r) e^{-i\theta} = (\frac{1}{r} + i\theta) e^{-i\theta} = \frac{e^{-i\theta}}{r}$

b) verify $u = x^2 - y^2 - y$ is harmonic find conjugate

$u_{xx} + u_{yy} = 0$

$u_x = 2x$, $u_{xx} = 2$, $u_y = -2y - 1$, $u_{yy} = -2$

$\nabla^2 u = 0$

$u_x = v_y \leftarrow \quad u_y = -v_x$

$v_y = u_x = 2x \longrightarrow \int v_y \, dy = \int 2x \, dy$

$v = 2xy + f(x) \longrightarrow v_x = 2y + f'(x) = 2y + 1$

$\therefore f'(x) = 1 \longrightarrow f(x) = x + c$

$\therefore v = 2xy + x + c$

c) $\ln \frac{(x+iy)}{(x-iy)} = 2i \tan^{-1}(\frac{y}{x})$

R.H.S $= \ln \frac{x+iy}{x-iy} = \ln \frac{re^{i\theta}}{re^{-i\theta}} = \ln e^{i\theta} e^{i\theta} = \ln e^{i2\theta}$

$= i2\theta = i2\tan^{-1}(y/x)$

d) $\int_{1+i}^{2+4i} z^2 \, dz$ along parabola $x = t$, $y = t^2$, $0 \le t < 2$

$dx = dt \quad dy = 2t\,dt$

$I = \int_{1+i}^{2+4i} (x+iy)(dx + i\,dy)$

$I = \int_{(1,1)}^{(2,4)} [(x^2 - y^2) + i2xy] (dx + i\,dy)$

$$= \int_{(1,1)}^{(2,4)} [(x^2-y^2)\,dx - (2xy)\,dy] + i\int_{(1,1)}^{(2,4)} (2xy)\,dx + (x^2-y^2)\,dy$$

$$I = \int_1^2 (t^2-t^4)\,dt - 2t^3(2t\,dt) + i\int_1^2 2t^3\,dt + (t^2-t^4)\,2t\,dt$$

$$I = \int_1^2 [(t^2-t^4-4t^4)\,dt + i\int_1^2 (4t^2-2t^5)\,dt = \frac{-86}{3} - i6$$

3 a) $c: |z| = 2$

i) $$\oint_c \frac{z^2-3}{z^2+4z+3}\,dz = \oint_c \frac{z^2-3}{(z+3)(z+1)}\,dz = \oint \frac{(z^2-3)/(z+3)}{(z+1)}\,dz.$$

$z_0 = -1$ inside c

$$I = 2\pi i\, f(z_0) = 2\pi i\left[\frac{(-1)^2-3}{-1+3}\right] = \boxed{-2\pi i}$$

ii) $$\oint \frac{e^z}{(z-1)^2}\,dz \Longrightarrow \oint_c \frac{f(z)}{(z-z_0)^{n+1}} = 2\pi i\, f'(z_0)$$

$z_0 = 1 \quad n = 1 \quad f(z) = e^z$

$$I = 2\pi i\,[e^1] = \boxed{2\pi e i}$$

iii) $$\oint_c \frac{\sin z}{(z+3)z^2}\,dz = \oint_c \frac{\sin z/(z+3)}{z^2}\,dz$$

$z_0 = 0$ inside $n = 1$

$$f'(z) = \frac{(z+3)\cos z - \sin z}{(z+3)^2}$$

$$I = \frac{2\pi i}{1!} f'(z_0) = 2\pi i\left[\frac{3}{9}\right] = \boxed{\frac{2}{3}\pi i}$$

b) if $f(z) = \ln(1+z)$ Find Maclaurin series of

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$f(z) = \ln(1+z) \qquad f(0) = \ln 1 = 0$$
$$f'(z) = (1+z)^{-1} \qquad f'(0) = 1$$
$$f''(z) = -(1+z)^{-2} \qquad f''(0) = -1$$
$$f'''(z) = 2(1+z)^{-3} \qquad f'''(0) = 2$$

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots$$
$$\ln(1-z) = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \cdots$$
$$\tanh^{-1} z = \frac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]$$
$$\tanh^{-1} z = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots$$

c) Find the order of the poles and find Residue at each pole $f(z) = \dfrac{z+1}{z^2(z-2)}$

$z=0$ is pole of order 2

$z=2$ " " " " 1 (simple pole)

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(1)!}\lim_{z\to 0} \frac{d}{dz} \cancel{(z^2)} \frac{z+1}{\cancel{z^2}(z-2)}$$
$$= \lim_{z\to 0} \frac{(z-2)-(z+1)}{(z-2)^2} = \frac{-3}{4}$$
$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to 2} \cancel{(z-2)} \frac{z+1}{z^2\cancel{(z-2)}} = \frac{3}{4}$$

d) $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta$

$z = e^{i\theta} \longrightarrow dz = ie^{i\theta} d\theta = iz\,d\theta \qquad d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{z+z^{-1}}{2} \qquad \cos 3\theta = \frac{z^3+z^{-3}}{2}$

$I = \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \oint_c \frac{\left(\frac{z^3+z^{-3}}{2}\right)}{\left[5-4\left(\frac{z+z^{-1}}{2}\right)\right]} * \frac{z^3}{z^3}$

$= \oint_c \frac{z^6 + i/2}{z^3[5-2z-2z^{-1}]} * \frac{dz}{iz}$

$= \frac{1}{2i} \oint_c \frac{z^6+1}{z^3(2z-1)(z-2)}$

(1) $z_0 = 0 \longrightarrow$ pole of order 3 $\longrightarrow$ inside C

(2) $z = \frac{1}{2} \longrightarrow$ " " " 1 $\longrightarrow$ " "

(3) $z = 2 \longrightarrow$ " " " 1 $\longrightarrow$ outside C

$R_1 = \operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m$

$= \frac{1}{2!} \lim_{z\to 0} \frac{d^3}{dz^3} z^3 \frac{z^6+1}{z^3(2z-1)(z-2)} = \frac{21}{8}$

$R_2 = \operatorname*{Res}_{z\to z_0} f(z) = \lim_{z\to 1/2} (z-1/2) \frac{z^6}{z^3(2z-1)(z-2)} = \frac{-65}{24}$

$I = \frac{-1}{2i}\left[2\pi i \sum \operatorname{Res} f(z)\right]$

$= \frac{1}{2i}(2\pi i)\left(\frac{21}{8} - \frac{65}{24}\right) = \frac{\pi}{12}$