نموذج إجابة

رياضيات هندسية ٣ ـ أ

تيرم أول

للفرقة الثانية

للعام الجامعي ٢٠١٧ ـ ٢٠١٨

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$Z_1 = 1 + i\,, Z_2 = +i\sqrt{3}, Z_3 = 1 - i\,, Z_4 = \sqrt{3} + i$

$Find\left(\frac{Z_1 Z_2 Z_3}{Z_4}\right), Arg\left(\frac{Z_1 Z_2}{Z_3 Z_4}\right)$

$$\frac{Z_1 Z_2 Z_3}{Z_4} = \frac{(1+i)\left(1+\sqrt{3}i\right)(1-i)}{\sqrt{3}+i} = \frac{\left(1+\sqrt{3}i + i - \sqrt{3}\right)(1-i)}{\sqrt{3}+i}$$

$$= \frac{2+2\sqrt{3}i}{\sqrt{3}+i} * \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{2\sqrt{3} - 2i + 6i + 2\sqrt{3}}{3-1}$$

$$= 2\sqrt{3} + 2i$$

$$\therefore \left(\frac{Z_1 Z_2 Z_3}{Z_4}\right) = 2\sqrt{3} + 2i$$

$$Arg\left(\frac{Z_1 Z_2}{Z_3 Z_4}\right) = Arg\,(Z_1 Z_2) - Arg\,(Z_3 Z_4)$$

$$= [ArgZ_1 + ArgZ_2] - [AgrZ_3 + ArgZ_4]$$

$$ArgZ_1 = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$$

$$ArgZ_2 = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$ArgZ_3 = \tan^{-1}\frac{-1}{1} = \frac{-\pi}{4}$$

$$ArgZ_4 = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$$\therefore Arg\left(\frac{Z_1 Z_2}{Z_3 Z_4}\right) = \left[\frac{\pi}{4} + \frac{\pi}{3}\right] - \left[\frac{-\pi}{4} + \frac{\pi}{6}\right]$$

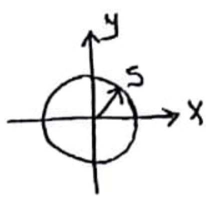
$$\therefore Arg\left(\frac{Z_1 Z_2}{Z_3 Z_4}\right) = \frac{2\pi}{3}$$

b) Find roots of the equation $e^z + 1 = 0$ which lie inside the $disc$ $|Z| = 4$

$e^z = 1 \quad r = \sqrt{(-1)^2 + 0} = 1 \quad \theta = \tan^{-1}\frac{0}{-1} = 0 + \pi$

$$\theta = \pi$$

$$e^z = re^{i\theta} = e^{i(\pi + 2n\pi)} \qquad n = 0, \pm 1, \pm 2, \ldots\ldots$$



$$at\ n = 0 \ \rightarrow Z = i(\pi + 2n\pi) = i\pi \ \rightarrow inside\ c$$

$$n = 1 \ \rightarrow Z = i\pi 3 = i3\pi \ \rightarrow outside\ c$$

$$n = -1 \rightarrow Z = -i\pi \rightarrow inside\ c$$

$$n = -2 \rightarrow Z = -i3\pi = i\pi \rightarrow outside\ c$$

$$Z = \{i\pi, -i\pi\}$$

1-c) $\lim_{z\to 1+i} \frac{Z^2-2Z+2}{Z^2-2i}$, $\quad z = 1+i \rightarrow x = 1, y = 1$

$$\lim_{z\to 1+i} \frac{(x+iy)^2 - 2(x+iy) + 2}{(x+iy)^2 - 2i}$$

$$\lim_{x\to 1}\lim_{y\to 1} \frac{(x+iy)^2 - 2(x+iy) + 2}{(x+iy)^2 - 2i} = \lim_{x\to 1} \frac{(x+i)^2 - 2(x+i) + 2}{(x+i)^2 - 2i} =$$

$$\lim_{x\to 1} \frac{2x + i2 - 2}{2x + i2} = \frac{i2}{2+i2} = \frac{-2}{i2-2}$$

$$\lim_{y\to 1}\lim_{x\to 1} \frac{(x+iy)^2 - 2(x+iy) + 2}{(x+iy)^2 - 2i} = \lim_{y\to 1} \frac{(1+iy)^2 - 2(1+iy) + 2}{(1+iy)^2 - 2i}$$

$$\lim_{y\to 1} \frac{1-y^2}{1 + i2y - y^2 - 2i} = \frac{0}{0}$$

$$\lim_{y\to 1} \frac{-2y}{i2 - 2y} = \frac{-2}{i2-2}$$

$$\therefore \lim_{z\to 1+i} \frac{Z^2 - 2Z + 2}{Z^2 - 2i} = \frac{-2}{i2-2}$$

1-D) Determine the following function is continuous at the given point:

$$f(Z) = \begin{cases} \frac{x^2 + iy^2}{|Z|} & Z \neq 0 \\ 1 & Z = 0 \end{cases} \quad at\ Z = 0$$

1) f(0) = 1 defined

2) $\lim_{z\to 0} \frac{x^2+iy^2}{|Z|^2} = \lim_{z\to 0} \frac{x^2+iy^2}{(\sqrt{x^2+y^2})^2}$

➔ $\lim_{x\to 0} \lim_{y\to 0} \frac{x^2+iy^2}{x^2+y^2} = \lim_{x\to 0} 1 = 1$

➔ $\lim_{y\to 0} \lim_{x\to 0} \frac{x^2+iy^2}{x^2+y^2} = \lim_{y\to 0} \frac{iy^2}{y^2} = i$

Limit is not exist

F(z) is discontinuous at Z = 0

1-e) Find the point of discontinuity of the function:

$$f(Z) = \frac{1}{Z} - \sec Z$$

$Z = \left\{0, (2n+1)\,{}^{\pi}/_{2}\right\}$ discontinuity points

(2) Discuss the differentiability of the following function:

$$W = \frac{1}{Z}$$

$$W = \frac{1}{Z} = \frac{1}{x - iy} * \frac{x+iy}{x+iy} = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$$

$Ux = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$	$Vx = \frac{-2yx}{(x^2+y^2)^2}$
$Uy = \frac{(x^2+y^2)0 - 2xy}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$	$Vy = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)}$ $= \frac{x^2-y^2}{(x^2+y^2)^2}$

$Ux \neq Vy$ ➔ C.R condition not satisfied.

$\therefore f(Z)$ has no differentiable

2-b) Prove the De Moiver's theorem, then use it to compute:

$$(1 + i)^6$$

$$Z = x + iy = r\,e^{i\theta} = r[\cos\theta + i\sin\theta]$$

$$Z^n = r^n\,e^{in\theta} = r^n[\cos\theta + i\sin\theta]^n$$

$$\therefore\ r^n[\cos\theta + i\sin\theta]^n = r^n[\cos n\theta + i\sin n\theta]$$

$$\therefore\ [\cos\theta + i\sin\theta]^n = [\cos n\theta + i\sin n\theta]$$

$$(1+i)^6 = \left[\sqrt{2}\left(\cos{}^{\pi}/_{4} + i\sin{}^{\pi}/_{4}\right)\right]^6$$

$$= \left(\sqrt{2}\right)^6\left(\cos 6\,{}^{\pi}/_{4} + i\sin 6\,{}^{\pi}/_{4}\right)^6$$

$$= 8\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = -8i$$

2-c) Determine whether the following function is harmonic or not, if your answer is yes, find conjugate harmonic and find $f^{`}(Z)$, $V = -e^{-x}\sin y$

$Vx = e^{-x}\sin y$ $\quad$ $Vy = -e^{-x}\cos\ y$

$Vxx = -e^{-x}\sin y$ $\quad$ $Vyy = -e^{-x}\sin\ y$

$Vxx + Vyy = 0$ ➔ $\therefore f(Z)$ is harmonic function

$$Uy = -Vx \ \rightarrow\ Uy = -e^{-x}\sin y$$

$$\int Uy\,dy = -\int e^{-x}\sin y\,dy \ \rightarrow\ U = e^{-x}\cos y + f(x)$$

$$Ux = Vy \ \rightarrow\ -e^{-x}\cos y + f^{`}(x) = -\,e^{-x}\cos y$$

$$\therefore f^{`}(x) = 0 \ \rightarrow\ \int f^{`}(x)\,dx = \int 0\,dx$$

$$\therefore f(x) = c$$

$$\therefore U = e^{-x}\cos y + c$$

$$f^{`}(x) = Vy - i\,Uy = -e^{-x}\cos y + i\,e^{-x}\cos y + c$$

2-d) Prove that not all harmonic function is analytic function for:

(3x – 2y) + i(x + y)

Ux = 3	Vx = 1
Uxx = 0	Vy = 0
Uy = -2	Vxx = 1
Uyy = 0	Vyy = 0

$\therefore Uxx + Uyy = 0$ ➔ U is harmonic

$\therefore Vxx + Vyy = 0$ ➔ V is harmonic

$\therefore f(Z)$ is harmonic function

$Ux \neq Vy$, $Uy \neq -Vx$ ➔ C.R condition not satisfied

$\therefore f(Z)$ is not analytic

3-a) By using Cauchy's Residue theorem to:

i) Show that $\int_c^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

$$Z = e^{i\theta} \rightarrow dZ = i\, e^{i\theta}\, d\theta \rightarrow d\theta = \frac{1}{i\, e^{i\theta}}\, dZ = \frac{dZ}{iZ}$$

$$\cos\theta = \frac{Z + Z^{-1}}{2} \quad , \quad \sin\theta = \frac{Z - Z^{-1}}{2i} \quad , \quad C : |Z| = 1$$

$$I = \oint_c \frac{1}{2 + \frac{Z + Z^{-1}}{2}} \frac{dZ}{iZ} * \frac{Z}{Z} = \frac{2}{i} \oint_c \frac{1}{(4 + Z + Z^{-1})\, Z}\, dZ$$

$$= \frac{2}{i}\oint_c \frac{1}{4Z + Z^2 + 1}\, dZ = \frac{2}{i}\oint_c \frac{1}{Z^2 + 4Z + 1}\, dZ$$

$$Z_{1,2} = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3} = -0.27 \quad -3.7$$

Z = -0.2 ➔ inside C , Z_2 = -3.7 outside C

$$R_1 = \underset{Z=-0.27}{Res} f(Z) = \lim_{Z\to -0.27} (Z+0.27) * \frac{1}{(Z+0.27)(Z+3.7)}$$

$$R_1 = \frac{1}{3.43} = 0.2915$$

$$\therefore I = \frac{2}{i}[2\pi\, i \underset{z=-0.27}{Res} f(z)] = 4\pi\,[0.2915] = \frac{2\pi}{\sqrt{3}}$$

3-a-ii) Evaluate: $\oint_{|Z|=2} \frac{1}{Z^4+5z^2+6}\ dz$

$$I = \int_c \frac{dZ}{(z^2+2)(z^2+3)}$$

$Z^2 = -2 \ \rightarrow Z = \pm i\sqrt{2}$ pole of order one (inside c)

$Z^2 = -3 \ \rightarrow Z = \pm i\sqrt{3}$ pole of order one (inside c)

$$R_1 = \underset{Z=Z_o}{Res} f(Z) = \lim_{Z\to Z_o} (Z - Z_o) f(Z)$$

At $Z = i\sqrt{2}$ ➔ $\underset{Z=i\sqrt{2}}{Res} f(Z) = \lim_{Z\to i\sqrt{2}} \left(Z - i\sqrt{2}\right) \frac{1}{(Z+i\sqrt{2})(Z-i\sqrt{2})(Z^2+3)}$

$$\boxed{R_1 = \frac{1}{2i\sqrt{2}}}$$

$R_2 = \underset{Z=-i\sqrt{2}}{Res} f(Z) = \lim_{Z\to -i\sqrt{2}} \left(Z + i\sqrt{2}\right) \frac{1}{(Z+i\sqrt{2})(Z-i\sqrt{2})(Z^2+3)}$

$$\boxed{R_2 = \frac{1}{-2i\sqrt{2}}}$$

$$R_3 = \underset{Z=i\sqrt{3}}{Res} f(Z) = \lim_{Z\to i\sqrt{3}} (Z + i\sqrt{3}) \frac{1}{(Z^2+2)(Z+i\sqrt{3})(Z-i\sqrt{3})}$$

$$\boxed{R_3 = \frac{1}{2i\sqrt{3}}}$$

$$R_4 = \underset{Z=-i\sqrt{3}}{Res} f(Z) = \lim_{Z\to -i\sqrt{3}} (Z + i\sqrt{3}) \frac{1}{(Z^2+2)(Z+i\sqrt{3})(Z-i\sqrt{3})}$$

$$\boxed{R_4 = \frac{1}{-2i\sqrt{3}}}$$

$$\therefore I = \oint \frac{dZ}{(Z^2+2)(Z^2+3)} = 2\pi \sum_{i=1}^{n} \underset{z=z_o}{Res} f(Z) = 2\pi\, i\, [0] = 0$$

3-b) Prove the Cauchy Integral formula, then use it to evaluate:

$$\oint_{|Z|=3} \frac{\sin \pi Z^2 + \cos \pi Z^2}{(Z-1)(Z-2)}\, dZ$$

$\frac{1}{(z-2)(z-1)} = \frac{1}{(z-2)} + \frac{-1}{(z-1)}$ by using partial fraction

$$I = \oint_c \frac{\sin \pi Z^2 + \cos \pi Z^2}{(Z-1)(Z-2)}\, dZ$$

$$= \left[\oint_c \frac{\sin \pi Z^2 + \cos \pi Z^2}{(z-2)} - \oint_c \frac{\sin \pi Z^2 + \cos \pi Z^2}{(z-1)}\right] dZ$$

$$I_1 at\ Z_o = 2 \;\rightarrow\; I_1 = \oint_c \frac{\sin \pi Z^2 + \cos \pi Z^2}{(z-2)}\, dZ = 2\pi i[\sin 4\pi + \cos 4\pi]$$

$$\boxed{I_1 = 2\pi i}$$

$$I_2 at\ Z_o = 2 \;\rightarrow\; I_2 = \oint_c \frac{\sin \pi Z^2 + \cos \pi Z^2}{(z-1)}\, dZ = 2\pi i[\sin \pi + \cos \pi]$$

$$\boxed{I_2 = -2\pi i}$$

$$\therefore I = I_1 - I_2 = 2\pi i - (-2\pi i) = 4\pi i$$

3-c) Expand $f(Z) = \frac{\sin z}{(Z-\pi)^2}$ in Laurent's near $Z = \pi$ and find the residue at this point.

Taylor series → $\sin Z = \sin\pi + (Z - \pi)\cos\pi - \frac{1}{2i}(Z - \pi)^2 \sin\pi$

$$-\frac{1}{3!}(Z-\pi)^3 \cos\pi + \ldots\ldots\ldots\ldots$$

$$\therefore \sin Z = -(Z - \pi) - \frac{1}{3!}(Z-\pi)^3 - \frac{1}{5!}(Z-\pi)^5 + \ldots\ldots\ldots$$

$$f(Z) = \frac{1}{(z-\pi)^2}\left[-(Z-\pi) - \frac{1}{3!}(Z-\pi)^3 - \frac{1}{5!}(Z-\pi)^5 + \ldots\ldots\ldots\right]$$

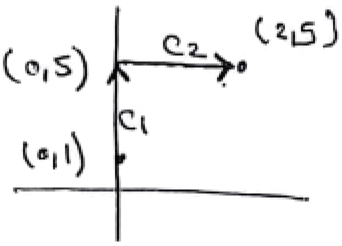
$$\boxed{f(Z) = \frac{-1}{Z-\mu} - \frac{1}{3!}(Z-\pi) - \frac{1}{5!}(Z-\pi)\ldots\ldots\ldots}$$

$$\boxed{\underset{Z=\pi}{Res} f(Z) = -1}$$

3-d) Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy$ along the straight line joining from (0, 1) to (0, 5) and from (0, 5) to (2, 5).

On $C_1 : x = 0 \ \rightarrow dx = 0$

$$I_{C_1} = \int_{y=1}^{y=5} (2y - 0)dy = [y^2]_1^5 = 25 - 1 = 24$$



On $C_2 : y = 5 \ \rightarrow dy = 0$

$$I_{C_1} = \int_{x=0}^{x=2} (3x+5)dx = \left[\frac{3x^2}{2} + 5x\right]_0^2 = \left(\frac{3(4)}{2} + 10\right) - 0 = 16$$

$$\boxed{I = I_{C_1} + I_{C_2} = 24 + 16 = 40}$$