نموذج إجابة

ثانية ك

Engineering Mathematics (3) A

Model Answer

2016 / 2017

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2nd year electrical Eng.

II

a) $z, w \in \mathbb{C}$

i) $\overline{z \pm w} = \bar{z} \pm \bar{w}$

let $z = a + ib$ & $w = c + id$

$\Rightarrow \overline{z \pm w} = \overline{(a+ib) \pm (c+id)} = \overline{(a \pm c) + i(b \pm d)}$

$= (a \pm c) - i(b \pm d) \rightarrow ①$

& $\bar{z} \pm \bar{w} = (a - ib) \pm (c - id) = (a \pm c) - i(b \pm d) \rightarrow ②$

from ① & ② $\Rightarrow \overline{z \pm w} = \bar{z} \pm \bar{w}$

ii) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

$\because \overline{\left(\frac{1}{z}\right)} = \overline{\left(\frac{1}{a+ib}\right)} = \overline{\left(\frac{a-ib}{a^2+b^2}\right)} = \frac{a+ib}{a^2+b^2} \rightarrow ①$

& $\frac{1}{\bar{z}} = \frac{1}{a-ib} = \frac{a+ib}{a^2+b^2} \rightarrow ②$

from ① & ② $\Rightarrow \overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

b) $z_1 = -4 + 4i$ & $z_2 = 3i$

(i) $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \tan^{-1}\frac{4}{-4} - \tan^{-1}\frac{3}{0} = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$

(ii) $|z_1 z_2| = |(-4+4i)(3i)| = |-12 - 12i| = \sqrt{144+144} = 3\sqrt{32} = 12\sqrt{2}$

(iii) $\operatorname{Im}\left(\frac{1}{z_1}\right) = \operatorname{Im}\left(\frac{1}{-4+4i}\right) = \operatorname{Im}\left(\frac{-4-4i}{32}\right) = \frac{-4}{32} = \frac{-1}{8}$

c) $z^2 = 3 + 4i \Rightarrow (x+iy)^2 = 3 + 4i \Rightarrow x^2 + 2ixy - y^2 = 3 + 4i$

$\Rightarrow x^2 - y^2 = 3 \rightarrow ①$ & $2xy = 4 \Rightarrow y = \frac{2}{x} \rightarrow ②$

At ① $\Rightarrow x^2 - \left(\frac{2}{x}\right)^2 = 3 \Rightarrow x^4 - 3x^2 - 4 = 0 \Rightarrow x^2 = \frac{3 \pm \sqrt{9+16}}{2}$

$\Rightarrow x^2 = 4$ or $x^2 = -1$ مرفوض

$\Rightarrow x = \pm 2 \Rightarrow y = \pm 1 \Rightarrow z = 2 + i$ or $z = -2 - i$

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d) $\because (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

let $n = 3$

$\Rightarrow (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$\Rightarrow \cos^3\theta + 3\cos^2\theta(i\sin\theta) + \frac{3*2}{2!}\cos\theta(i\sin\theta)^2 + \frac{3*2*1}{3!}(i\sin\theta)^3$

$= \cos 3\theta + i\sin 3\theta$

$\Rightarrow (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta$

$\because$ Real = Real & Im = Im

$\Rightarrow \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

& $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

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$\boxed{2}$ a) $f(z) = \begin{cases} z^2 & , z \neq -i \\ 0 & , z = -i \end{cases}$

$\because$ (1) $f(-i) = 0$

(2) $\lim_{z\to -i} f(z) = \lim_{z\to -i} z^2 = (-i)^2 = -1$

$\therefore$ (3) $\lim_{z\to -i} f(z) \neq f(-i)$

$\Rightarrow f(z)$ is discontinuous at $z = -i$

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b) $\frac{d}{dz} z^n = n\, z^{(n-1)}$

let $z = re^{i\theta} \Rightarrow f(z) = z^n = r^n e^{in\theta} = r^n[\cos n\theta + i\sin n\theta]$

$\Rightarrow f(z) = r^n\cos n\theta + i\, r^n\sin n\theta = u + iv$

$\Rightarrow u = r^n\cos n\theta$ & $v = r^n\sin n\theta$

$\Rightarrow u_r = nr^{n-1}\cos n\theta$ & $v_r = nr^{n-1}\sin n\theta$

$u_\theta = -nr^n\sin n\theta$ & $v_\theta = nr^n\cos n\theta$

$\gg$ $u_r = \frac{1}{r}v_\theta$, $v_r = \frac{-1}{r}u_\theta$ > C.R.C satisfied

$\Rightarrow f'(z) = (u_r + iv_r)e^{-i\theta} = (nr^{n-1}\cos n\theta + i\,nr^{n-1}\sin n\theta)e^{-i\theta}$

$\Rightarrow f'(z) = nr^{n-1}[\cos n\theta + i\sin n\theta]e^{-i\theta} = nr^{n-1}e^{in\theta}e^{-i\theta} = nr^{n-1}e^{i\theta(n-1)} = n\,z^{n-1}$

c) $\because u = 3x^2y + 2x^2 - y^3 - 2y^2$

$\Rightarrow u_x = 6xy + 4x \quad \& \quad u_y = 3x^2 - 3y^2 - 4y$

$u_{xx} = 6y + 4 \qquad u_{yy} = -6y - 4$

$\because u_{xx} + u_{yy} = \text{zero} \Rightarrow u$ is harmonic $f^{n}$

From C.R.C.

$\Rightarrow \& \ u_x = v_y \Rightarrow v_y = 6xy + 4x \Rightarrow v = \int (6xy + 4x)\,dy$

$\Rightarrow v = 3xy^2 + 4xy + f(x)$

$\& \because v_x = -u_y \Rightarrow v_x = -3x^2 + 3y^2 + 4y$

$\Rightarrow \cancel{3y^2} + \cancel{4y} + f'(x) = -3x^2 + \cancel{3y^2} + \cancel{4y}$

$\Rightarrow f'(x) = -3x^2 \Rightarrow f(x) = -x^3 + c$

$\Rightarrow v = 3xy^2 + 4xy - x^3 + c$ is a conjugate harmonic of $u$

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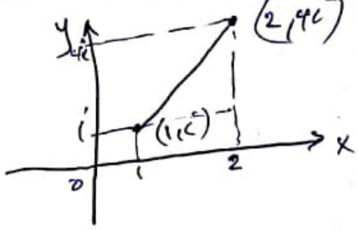
d) $\int_{1+i}^{2+4i} z\,dz$




i) $x = t, \quad y = t^2, \quad 1 < t < 2$

$I = \int_{1+i}^{2+4i} (x+iy)(dx + i\,dy) = \int_{1+i}^{2+4i} (x+iy)\,dx + i(x+iy)\,dy$

at $x = t \Rightarrow dx = dt \ \& \ y = t^2 \Rightarrow dy = 2t\,dt$

$\Rightarrow I = \int_1^2 (t + it^2)\,dt + i(t + it^2)(2t\,dt)$

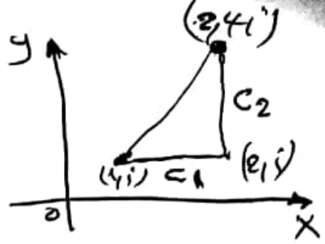
$= \int_1^2 (t + it^2 + 2it^2 - 2t^3)\,dt$

$= \frac{t^2}{2} + i\frac{t^3}{3} + \frac{2}{3}it^3 - \frac{t^4}{2} \Big|_1^2$

$= 2 + \frac{8}{3}i + \frac{16}{3}i - 8 - \cancel{\frac{1}{2}} - \frac{1}{3}i - \frac{2}{3}i + \cancel{\frac{1}{2}}$

$= -6 + 7i$

ii) along the straight line $1+i$ to $2+i$ and from $2+i$ to $2+4i$




on $C_1$; $1 < x < 2$ & $y = 1 \Rightarrow dy = 0$

$$\therefore I_{c_1} = \int_1^2 (x+i)\,dx = \left[\frac{x^2}{2} + ix\right]\Big|_1^2 = 2 + 2i - \frac{1}{2} - i$$

$$= \frac{3}{2} + i \rightarrow (1)$$

on $C_2$; $1 < y < 4$ & $x = 2 \Rightarrow dx = 0$

$$\therefore I_{c_2} = \int_1^4 i(2+iy)\,dy = i\left(2y + i\frac{y^2}{2}\right)\Big|_1^4$$

$$= i\left[8 + i8 - 2 - i\frac{1}{2}\right] = i\left[6 + \frac{15}{2}i\right]$$

$$= -\frac{15}{2} + 6i \rightarrow (2)$$

from (1) & (2)

$$\Rightarrow I = I_{c_1} + I_{c_2}$$

$$= \frac{3}{2} + i - \frac{15}{2} + 6i = -6 + 7i$$

$\boxed{3}$ a) $C: |z| = 2$

i) $\oint_c \sin z \, dz$

$\because f(z) = \sin z$ is analytic $f^{\underline{n}}$ on and inside $C$

$\Rightarrow \oint_c \sin z \, dz = \text{zero}$

ii) $\oint_c \frac{6e^z}{(z-1)} dz$

$\because f(z) = 6e^z$ is analytic $f^{\underline{n}}$ on and inside $C$

& $z-1=0 \Rightarrow z_0 = 1$

$\therefore \oint_c \frac{6e^z}{z-1} dz = 2\pi i \, f(z_0) = 2\pi i \, (6e^1) = 12\pi e i$

iii) $\oint_c \frac{\cos z}{(z+4) z^2} dz$

$\because f(z) = \frac{\cos z}{z+4}$ is analytic $f^{\underline{n}}$ on and inside $C$

& $z^2 = 0 \Rightarrow z_0 = 0$ is a repeated pole inside $C$

$\Rightarrow f'(z) = \frac{(z+4)\sin z - \cos z}{(z+4)^2}$

$\Rightarrow f'(z_0) = \frac{-1}{16}$

$\Rightarrow \oint_c \frac{\cos z}{(z+4) z^2} dz = 2\pi i \, f'(z_0) = \frac{-2\pi i}{16} = \frac{-\pi i}{8}$

b) $f(z) = \dfrac{1}{1-z^2}$

$\because \dfrac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \alpha^3 + \text{---} \quad , \alpha < 1$

$\Rightarrow f(z) = \dfrac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \text{---} \quad , z^2 < 1$

is a Maclaurine series of $\dfrac{1}{1-z^2}$

$\because \dfrac{d}{dz} \coth^{-1} z = \dfrac{1}{1-z^2}$

$\Rightarrow \coth^{-1} z = \int \left(\dfrac{1}{1-z^2}\right) dz$

$= \int [1 + z^2 + z^4 + z^6 + \text{---}] dz$

$= z + \dfrac{z^3}{3} + \dfrac{z^5}{5} + \dfrac{z^7}{7} + \text{---} + c$

& $c = 0$ where $\coth^{-1} 0 = 0$

c) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

poles: $z = -1$ is a pole of order 2

$\because z^2 + 4 = 0 \Rightarrow z = \pm 2i \Rightarrow \begin{cases} z = 2i \text{ is a pole of order } 1 \\ z = -2i \text{ is a pole of order } 1 \end{cases}$

$\because \operatorname*{Res}_{z=z_0} f(z) = \dfrac{1}{(m-1)!} \lim_{z \to z_0} \dfrac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z)$ where m is the order of pole

At $z = -1 \Rightarrow \operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} \dfrac{d}{dz} \cancel{(z+1)^2} \dfrac{z^2 - 2z}{\cancel{(z+1)^2}(z^2+4)}$

$= \lim_{z \to -1} \dfrac{(z^2+4)(2z-2) - (z^2-2z)(2z)}{(z^2+4)^2} = \dfrac{-14}{25}$

At $z_2 = 2i$

$$\Rightarrow \operatorname*{Res}_{z=2i} f(z) = \lim_{z \to 2i} \cancel{(z-2i)} \frac{z^2-2z}{(z+1)^2\cancel{(z-2i)}(z+2i)} = \frac{-4-4i}{(2i+1)^2(4i)}$$

$$= \frac{-1-i}{(-4+4i+1)i} = \frac{-1-i}{(-4-3i)} * \frac{-4+3i}{-4+3i} = \frac{4+4i-3i+3}{25}$$

$$= \frac{7}{25} + \frac{1}{25}i$$

At $z_3 = -2i$

$$\Rightarrow \operatorname*{Res}_{z=-2i} f(z) = \lim_{z \to -2i} \cancel{(z+2i)} \frac{z^2-2z}{(z+1)^2(z-2i)\cancel{(z+2i)}}$$

$$= \frac{7}{25} - \frac{1}{25}i$$

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d) $$I = \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$$

let $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$

& $\cos\theta = \frac{z^1+z^{-1}}{2} \Rightarrow \cos 2\theta = \frac{z^2+z^{-2}}{2}$

$$\Rightarrow I = \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta = \oint_c \frac{\left(\frac{z^2+z^{-2}}{2}\right)}{\left[5-4\left(\frac{z+z^{-1}}{2}\right)\right]} \frac{dz}{iz}$$

$$= \frac{1}{2i}\oint_c \frac{z^4+1}{z^2\{5z-2z^2-2\}} dz = \frac{-1}{2i} \oint_c \frac{z^4+1}{z^2(2z^2-5z+2)} dz$$

$$= \frac{-1}{2i} \oint_c \frac{z^4+1}{z^2(2z-1)(z-2)} dz$$

Then $z_1 = 0$ is a pole of order 2 inside $c$

$z_2 = \frac{1}{2}$ " " " " 1 " "

$z_3 = 2$ " " " " 1 outside $c$

at $z_1 = 0$

$$\Rightarrow R_1 = \operatorname*{Res}_{z=0} f(z) = \left(\frac{-1}{2i}\right) \lim_{z \to 0} \frac{d}{dz} \cancel{z^2} \frac{z^4+1}{\cancel{z^2}(2z-1)(z-2)}$$

$$= \frac{-1}{2i} \lim_{z \to 0} \frac{-(4z-5)(z^4+1) + (2z^2-5z+2)(4z^3)}{[(2z-1)(z-2)]^2}$$

$$= \frac{-1}{2i} \left\{ \frac{+5}{4} \right\} = \frac{-5}{8i} = \frac{+5}{8} i$$

at $z_2 = \frac{1}{2}$

$$\Rightarrow R_2 = \operatorname*{Res}_{z=\frac{1}{2}} f(z) = \left(\frac{-1}{2i}\right) \lim_{z \to \frac{1}{2}} \frac{z^4+1}{z^2 [\cancel{2z-1}][z-2]} \cancel{[2z-1]}$$

$$= \left(\frac{-1}{2i}\right) \frac{\frac{1}{16}+1}{\frac{1}{4}\left(\frac{1}{2}-2\right)} = \frac{-17}{-12i} = \frac{-17}{12} i$$

$$\Rightarrow I = 2\pi i \{R_1 + R_2\}$$

$$= 2\pi i \left[ \frac{+5}{8} i - \frac{17}{12} i \right]$$

$$= \cancel{2}\pi \left[ \frac{-15+34}{\cancel{24}\ 12} \right] = \frac{19\pi}{12}$$