Year : Preparatory year 1/6/2019

Q3 : 25 Marks

$\boxed{1}$ a.) $I = \int \frac{x^4 - 4}{x^3 - 3x^2 - x + 3} dx$

$= \int \left[x + 3 + \frac{3}{4} \frac{1}{x-1} + \frac{77}{8} \cdot \frac{1}{x-3} - \frac{3}{8} \frac{1}{x+1} \right] dx$

$= \frac{x^2}{2} + 3x + \frac{3}{4} \ln|x-1| + \frac{77}{8} \ln|x-3| - \frac{3}{8} \ln|x+1| + C$

b) $I = \int e^{3x} \sqrt{1 - e^{2x}} \, dx$

$1 - e^{2x} = t^2 \Rightarrow -2e^{2x} dx = 2t\,dt$

$\therefore I = \int t \sqrt{1-t^2} \, (-t\,dt) = \int -t^2 \sqrt{1-t^2} \, dt$

$t = \sin\theta$

$dt = \cos\theta \, d\theta$

$I = -\int \sin^2\theta \cos^2\theta \, d\theta$

$= -\frac{1}{8} \int [1 - \cos 2\theta][1 + \cos 2\theta] \, d\theta$

$= -\frac{1}{8} \int [1 - \cos^2 2\theta] \, d\theta$

$= -\frac{1}{8} \int \left[1 - \frac{1}{2}(1 + \cos 4\theta)\right] d\theta$

b $I = -\frac{1}{8} \int \left[1 - \frac{1}{2} - \frac{1}{2} \cos 4\theta \right] d\theta$

$$= \frac{-1}{8} \left[\frac{\theta}{2} - \frac{1}{8} \sin 4\theta \right] + C$$

$$\theta = \sin^{-1} t = \sin^{-1} \sqrt{1 - e^{2x}}$$

$$\sin 4\theta = \frac{1}{2} \sin 2\theta \cos 2\theta = \frac{1}{2} \sin 2\theta (1 - 2\sin^2\theta$$

$$= \frac{1}{2} \sin\theta \cos\theta (1 - 2\sin^2\theta)$$

$$= \frac{1}{2} (\sin\theta - \sin^3\theta) \cos\theta$$

$$) \ I = \int \sqrt{\frac{x}{x-1}}\, dx \ * \frac{\sqrt{x}}{\sqrt{x}}$$

$$= \frac{1}{2} \int \frac{2x - 1 + 1}{\sqrt{x^2 - x}}\, dx =$$

$$= \int \frac{2x-1}{\sqrt{x^2-x}}\, dx + \frac{1}{2} \int \frac{1}{\sqrt{(x-\frac{1}{2})^2 - \frac{1}{4}}}\, dx$$

$$= \frac{1}{2} \frac{(x^2 - x)^{1/2}}{1/2} + \frac{1}{2} \cosh^{-1} \frac{(x - \frac{1}{2})}{\frac{1}{2}} + C$$

$$d) \ I = \int \ln (1 + \sin x)^{\sin 2x}\, dx$$

$$= \int \sin 2x \ \ln (1 + \sin x)\, dx$$

$$= \int 2 \sin x \cos x \ \ln (1 + \sin x)\, dx$$

let $t = \sin x$

$dt = \cos x \, dx$

$$I = 2 \int t \ \ln (1 + t)\, dt$$

$u = \ln(1+t)$ $\qquad dv = t\, dt$

$du = \frac{1}{1+t}\, dt$ $\qquad v = \frac{t^2}{2}$

$$I = \cancel{2t} \ \ln (1+t) - \int \frac{t^2}{1+t}\, dt$$

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b $$I = t^2 \ln(1+t) - \int \left[t - 1 + \frac{1}{t+1}\right] dt$$

$$= t^2 \ln(1+t) - \frac{t^2}{2} + t - \ln|t+1| + C$$

$$= \sin^2 x \ln(1+\sin x) - \frac{\sin^2 x}{2} + \sin x$$

$$- \ln|1+\sin x| + C$$

$\boxed{2}$ $$I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$$

$u = \sin^{n-1} x$ $\qquad$ $dV = \sin x \, dx$

$du = (n-1) \sin^{n-2} x \cos x \, dx$ $\qquad$ $V = -\cos x$

$$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) d$$

$$\therefore I_n = \frac{1}{n}\left[-\cos x \sin^{n-1} x + (n-1) I_{n-2}\right]$$

$n = 2, 3, \ldots$

For $n = 3$

$$I_3 = \frac{1}{3}\left[-\cos x \sin^2 x + 2 I_1\right]$$

$$I_1 = \int \sin x \, dx = -\cos x + C$$

$$\therefore I_3 = \frac{1}{3}\left[-\cos x \sin^2 x - 2 \cos x\right] + C$$

$$y = \ln \sec x$$

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x$$

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\pi/4} \sec x \, dx = \ln(\sec x + \tan x) \Big|_0^{\pi/4}$$

$$= \ln(\sqrt{2} + 1) - \ln(\cancel{1}) \rightarrow 0$$

b. Q4:-

$\boxed{1}$ $I = \int_1^2 \frac{1}{1+x}\,dx = \ln|1+x|\Big|_1^2$

$= \ln 3 - \ln 2 = exa$

X	1	1.2	1.4	1.6	1.8	2
y						

$h = \frac{2-1}{5} = 0.2$

using Trapezoidal method

$\int_1^2 \frac{1}{1+x}\,dx \simeq \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + y_4\right.$

error = exact - approximate

$\boxed{2}$ a) $\displaystyle\int_{-1}^{1} \frac{(\tan^{-1}x)^2 + \sin^3 x}{1+x^2}\, dx$

$$I = \int_{-1}^{1} \frac{(\tan^{-1}x)^2}{1+x^2}\, dx + \underbrace{\int_{-1}^{1} \frac{\sin^3 x}{1+x^2}\, dX}_{\to 0 \;\; \text{odd}}$$

$$\therefore I = 2\int_{0}^{1} \frac{(\tan^{-1}x)^2}{1+x^2}\, dx$$

$$= 2\, \frac{(\tan^{-1}x)^3}{3}\Big|_0^1$$

$$= \frac{2}{3}\left[(\tan^{-1}1)^3 - (\tan^{-1}0)^3\right]$$

$$= \frac{2}{3}\left(\frac{\pi}{4}\right)^3 = \frac{\pi^3}{96}$$

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b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 2\, \cancel{\tan^{-1}}$

$$= 2\int_{0}^{\infty} \frac{1}{1+x^2}\, dx = 2\lim_{a\to\infty}\int_0^a \frac{1}{1+x^2}\, dx$$

$$= 2\lim_{a\to\infty} \tan^{-1}x \Big|_0^a$$

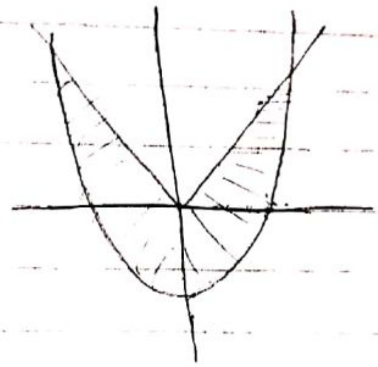
$$= 2\lim_{a\to\infty}\left[\tan^{-1}a - \underbrace{\tan^{-1}0}_{\to 0}\right] = \pi$$

b)

3)

$y_1 = x^2 - 3$

$y_2 = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$



$x^2 - 3 = 2x$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

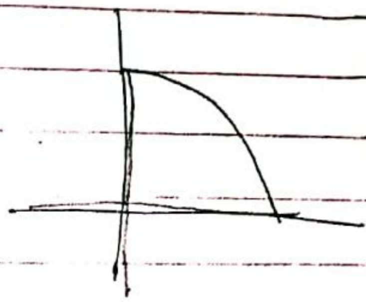
$\therefore x = 3$ or $x = -1 \rightarrow$ refused

$$\text{Area} = 2\int_0^3 (2x - (x^2-3))\,dx$$

$$= 2\left[x^2 - \frac{x^3}{3} + 3x\right]_0^3$$

$$= 2\left[9 - \frac{27}{3} + 9\right] = 18$$

1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$$\text{Volume} = \pi \int x^2\,dy$$

$$= \pi \int_0^b a^2\left[1-\frac{y^2}{b^2}\right] dy$$

$$= \pi a^2 \left[y - \frac{y^3}{3b^3}\right]\Big|_0^b$$

$$= \pi a^2\left[b - \frac{b}{3}\right] = \frac{2}{3}\pi a^2 b \rightarrow (1)$$

For The sphere $a = b$

$\therefore$ From (1) The volume of semisphere is $\frac{2}{3}\pi a^3$

$\therefore$ The volume of the sphere $= \frac{4}{3}\pi a^3$