نموذج إجابة

رياضيات هندسية ٣-أ

ترم أول

للفرقة الثانية ك

للعام الجامعي 2017-2018

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①

حل نموذج إجابة رياضيات هندسية ٣

[1] a- let $z_1 = 1+i$, $z_2 = 1+i\sqrt{3}$, $z_3 = 1-i$, $z_4 = \sqrt{3}+i$

Find $\overline{\left(\frac{z_1 z_2 z_3}{z_4}\right)}$, $Arg\left(\frac{z_1 z_2}{z_3 z_4}\right)$

$$\frac{z_1 z_2 z_3}{z_4} = \frac{(1+i)(1+\sqrt{3}i)(1-i)}{\sqrt{3}+i} = \frac{(1+\sqrt{3}i+i-\sqrt{3})(1-i)}{\sqrt{3}+i}$$

$$= \frac{2+2\sqrt{3}i}{\sqrt{3}+i} * \frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{2\sqrt{3}-2i+6i+2\sqrt{3}}{3-1} = \frac{4\sqrt{3}+4i}{2}$$

$$= 2\sqrt{3}+2i$$

$$\therefore \overline{\left(\frac{z_1 z_2 z_3}{z_4}\right)} = \boxed{2\sqrt{3}-i2}$$

$$Arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = Arg(z_1 z_2) - Arg(z_3 z_4)$$

$$= [Arg\, z_1 + Arg\, z_2] - [Arg\, z_3 + Arg\, z_4]$$

$\because$

$$Arg\, z_1 = \tan^{-1}\frac{1}{1} = \pi/4$$

$$Arg\, z_2 = \tan^{-1}\frac{\sqrt{3}}{1} = \pi/3$$

$$Arg\, z_3 = \tan^{-1}\frac{-1}{1} = -\pi/4$$

$$Arg\, z_4 = \tan^{-1}\frac{1}{\sqrt{3}} = \pi/6$$

$$\therefore Arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = [\pi/4 + \pi/3] - [-\pi/4 + \pi/6]$$

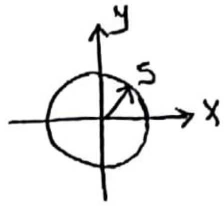
$$\therefore Arg\left(\frac{z_1 z_2}{z_3 z_4}\right) = \boxed{\frac{2\pi}{3}}$$

②

-b) Find roots of the equation $e^{Z}+1=0$ which lie inside the disc $|Z|=4$

$e^{Z}=-1$ $\quad r=\sqrt{(-1)^2+0}=1$ $\quad \theta=\tan^{-1}\frac{0}{-1}=0+\pi$

$\theta=\pi$



$e^{Z}=re^{i\theta}=e^{i(\pi+2n\pi)}$ $\quad n=0,\pm1,\pm2,\ldots$

at $n=0 \rightarrow Z=i(\pi+2n\pi)=i\pi \longrightarrow$ inside C

$n=1 \longrightarrow Z=i\,\pi 3=i3\pi \longrightarrow$ outside C

$n=-1 \longrightarrow Z=-i\pi \longrightarrow$ inside C

$n=-2 \longrightarrow Z=-i3\pi \longrightarrow$ outside C

$$\boxed{Z=\{i\pi,\ -i\pi\}}$$

1-c) $\displaystyle\lim_{Z\to 1+i}\frac{Z^2-2Z+2}{Z^2-2i}$,

$Z=1+i \longrightarrow X=1,\ y=1$

$$\lim_{Z\to 1+i}\frac{(X+iy)^2-2(X+iy)+2}{(X+iy)^2-2i}$$

$$\lim_{X\to 1}\lim_{y\to 1}\frac{(X+iy)^2-2(X+iy)+2}{(X+iy)^2-2i}=\lim_{X\to 1}\frac{(X+i)^2-2(X+i)+2}{(X+i)^2-2i}$$

$$=\lim_{X\to 1}\frac{2X+i2-2}{2X+i2}=\frac{i2}{2+i2}*\frac{i}{i}=\boxed{\frac{-2}{i2-2}}$$

$$\lim_{y\to 1}\lim_{X\to 1}\frac{(X+iy)^2-2(X+iy)+2}{(X+iy)^2-2i}=\lim_{y\to 1}\frac{(1+iy)^2-2(1+iy)+2}{(1+iy)^2-2i}$$

$$\lim_{y\to 1}\frac{1-y^2}{1+i2y-y^2-2i}=\frac{0}{0}$$

$$=\lim_{y\to 1}\frac{-2y}{i2-2y}=\boxed{\frac{-2}{i2-2}}$$

$$\therefore \lim_{Z\to 1+i}\frac{Z^2-2Z+2}{Z^2-2i}=\boxed{\frac{-2}{i2-2}}$$

③

1-d) determine the following function is continuous at the given point $f(z) = \begin{cases} \dfrac{x^2+iy^2}{|z|^2} & z \neq 0 \\ 1 & z = 0 \end{cases}$ at $z=0$

1) $f(0)=1$ defined

2) $\lim\limits_{z\to 0} \dfrac{x^2+iy^2}{|z|^2} = \lim\limits_{z\to 0} \dfrac{x^2+iy^2}{(\sqrt{x^2+y^2})^2}$

$\Rightarrow \lim\limits_{x\to 0} \lim\limits_{y\to 0} \dfrac{x^2+iy^2}{x^2+y^2} = \lim\limits_{x\to 0} 1 = 1$

$\Rightarrow \lim\limits_{y\to 0} \lim\limits_{x\to 0} \dfrac{x^2+iy^2}{x^2+y^2} = \lim\limits_{y\to 0} \dfrac{iy^2}{y^2} = i$

limit is not exist

$f(z)$ is discontinuous at $z=0$

~~1-e) Discuss the differentiability of the following function~~

1-e) Find the point of discontinuity of the function

$f(z) = \dfrac{1}{z} - \sec z$

where $n = 0, \pm 1, \pm 2, \dots$

$z = \{0, (2n+1)\pi/2\}$ discontinuity points

$\boxed{2}$ a- Discuss the differentiability of the following function

$w = \dfrac{1}{z}$

$w = \dfrac{1}{z} = \dfrac{1}{x-iy} * \dfrac{x+iy}{x+iy} = \dfrac{x+iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} + i \dfrac{y}{x^2+y^2}$

$u_x = \dfrac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$ | $v_x = \dfrac{-2yx}{(x^2+y^2)^2}$

$v_y = \dfrac{(x^2+y^2)-2y^2}{(x^2+y^2)} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$u_y = \dfrac{\cancel{(x^2+y^2)}0-2xy}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$

$u_x \neq v_y \Longrightarrow$ C.R condition not satisfied $\therefore f(z)$ not analytic

$\therefore f(z)$ has no differentiable

2-b) Prove the De Moiver's theorem, then use it to compute $(1+i)^6$

$Z = x+iy = re^{i\theta} = r\,[\cos\theta + i\sin\theta]$

$Z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n\,[\cos\theta + i\sin\theta]^n$

$\therefore r^n\,[\cos\theta + i\sin\theta]^n = r^n\,[\cos n\theta + i\sin n\theta]$

$[\cos\theta + i\sin\theta]^n = [\cos n\theta + i\sin n\theta]$ #

$(1+i)^6 = [\sqrt{2}\,(\cos \pi/4 + i\sin \pi/4)]^6$

$= (\sqrt{2})^6\,(\cos 6\,\pi/4 + i\sin 6\,\pi/4)$

$= 8\,[\cos \frac{3\pi}{2} + i\sin\frac{3\pi}{2}] = -8i$

2-c) Determine whether the following function is harmonic or not If your answer is yes, find conjugate harmonic and find $f'(z)$, $V = -e^{-x}\sin y$

$v_x = e^{-x}\sin y$ $\quad$ $v_y = -e^{-x}\cos y$

$v_{xx} = -e^{-x}\sin y$ $\quad$ $v_{yy} = e^{-x}\sin y$

$v_{xx} + v_{yy} = 0 \longrightarrow \therefore f(z)$ is harmonic function

$\therefore u_y = -v_x \longrightarrow u_y = -e^{-x}\sin y$

$\int u_y\,dy = -\int e^{-x}\sin y\,dy \longrightarrow u = e^{-x}\cos y + f(x)$

$u_x = v_y \longrightarrow -e^{-x}\cancel{\cos y} + f'(x) = -e^{-x}\cancel{\cos y}$

$\therefore f'(x) = 0 \longrightarrow \int f'(x)\,dx = \int 0\,dx$

$\therefore f(x) = c$

$\therefore u = e^{-x}\cos y + c$

$f'(x) = v_y + i u_y = -e^{-x}\cos y + i e^{-x}\cos y + c$

(5)

2-d) prove that, not all harmonic function is analytic function for $(3x-2y)+i(x+y)$

$u_x = 3$ $\qquad v_x = 1$

$u_{xx} = 0$ $\qquad v_y = 0$

$u_y = -2$ $\qquad v_{xx} = 1$

$u_{yy} = 0$ $\qquad v_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0 \longrightarrow u$ is harmonic

$v_{xx} + v_{yy} = 0 \longrightarrow v$ " "

$\therefore f(z)$ is harmonic function

$u_x \neq v_y$ $\quad$ & $\quad u_y \neq -v_x \longrightarrow$ C.R condition not satisfied

$\therefore f(z)$ is not analytic

3-a) By using cauchy's Residue theorem to

i) show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

$z = e^{i\theta} \longrightarrow dz = i e^{i\theta} d\theta \longrightarrow d\theta = \frac{1}{ie^{i\theta}} dz = \frac{dz}{iz}$

$\cos\theta = \frac{z+z^{-1}}{2}$, $\quad \sin\theta = \frac{z-z^{-1}}{2i}$, $\quad C: |z| = 1$

$$I = \oint_C \frac{1}{2+\frac{z+z^{-1}}{2}} \frac{dz}{iz} * \frac{z}{z} = \frac{2}{i} \oint_C \frac{1}{(4+z+z^{-1})z} dz$$

$$= \frac{2}{i} \oint_C \frac{1}{4z+z^2+1} dz = \frac{2}{i} \oint_C \frac{1}{z^2+4z+1} dz$$

$z_{1,2} = \frac{-4 \pm \sqrt{16-4}}{2} = \boxed{-2 \pm \sqrt{3}} = \begin{cases} -0.27 \\ -3.7 \end{cases}$

$z = -0.2 \longrightarrow$ inside c $\quad$, $z_2 = -3.7$ outside c

$$R_1 = \underset{z=-0.27}{\text{Res}} f(z) = \lim_{z \to -0.27} \cancel{(z+0.27)} * \frac{1}{\cancel{(z+0.27)}(z+3.7)}$$

$R_1 = \frac{1}{3.43} = 0.2915$

$$\therefore I = \frac{2}{\cancel{i}} [2\pi \cancel{i} \underset{z=-0.27}{\text{Res}} f(z)] = 4\pi[0.2915] = \frac{2\pi}{\sqrt{3}}$$

(6)

3-a-ii) Evaluate $\oint_{|z|=2} \frac{1}{z^4+5z^2+6} dz$

$$I = \oint_c \frac{dz}{(z^2+2)(z^2+3)}$$

$z^2 = -2 \longrightarrow z = \pm i\sqrt{2}$ pole of order one (inside c)

$z^2 = -3 \longrightarrow z = \pm i\sqrt{3}$ pole of order one (inside c)

$$R_1 = \operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

at $z = i\sqrt{2} \longrightarrow R_1 = \operatorname{Res}_{z=i\sqrt{2}} f(z) = \lim_{z \to i\sqrt{2}} \cancel{(z-i\sqrt{2})} \frac{1}{(z+i\sqrt{2})\cancel{(z-i\sqrt{2})}(z^2+3)}$

$$\boxed{R_1 = \frac{1}{2i\sqrt{2}}}$$

$$R_2 = \operatorname{Res}_{z=-i\sqrt{2}} f(z) = \lim_{z \to -i\sqrt{2}} \cancel{(z+i\sqrt{2})} \frac{1}{\cancel{(z+i\sqrt{2})}(z-i\sqrt{2})(z^2+3)}$$

$$\therefore \boxed{R_2 = \frac{1}{-2i\sqrt{2}}}$$

$$R_3 = \operatorname{Res}_{z=i\sqrt{3}} f(z) = \lim_{z \to i\sqrt{3}} \cancel{(z-i\sqrt{3})} \frac{1}{(z^2+2)(z+i\sqrt{3})\cancel{(z-i\sqrt{3})}}$$

$$\boxed{R_3 = \frac{1}{2i\sqrt{3}}}$$

$$R_4 = \operatorname{Res}_{z=-i\sqrt{3}} f(z) = \lim_{z \to -i\sqrt{3}} \cancel{(z+i\sqrt{3})} \frac{1}{(z^2+2)\cancel{(z+i\sqrt{3})}(z-i\sqrt{3})}$$

$$\boxed{R_4 = \frac{1}{-2i\sqrt{3}}}$$

$$\therefore I = \oint \frac{dz}{(z^2+2)(z^2+3)} = 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{z=z_i} f(z)$$

$$\therefore = 2\pi i \,[0] = \boxed{0}$$

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3-b) prove the Cauchy Integral formula, then use it to Evaluate $\oint_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

$$\frac{1}{(z-2)(z-1)} = \frac{1}{(z-2)} + \frac{-1}{(z-1)}$$ by using partial fraction

$$I = \oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-1)} dz = \left[\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} - \oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}\right] dz$$

I_1 at $z_0 = 2 \rightarrow I_1 = \oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} dz = 2\pi i [\sin 4\pi + \cos 4\pi] = \boxed{2\pi i}$

I_2 at $z_0 = 1 \longrightarrow I_2 = \oint \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$

$= 2\pi i [\sin \pi + \cos \pi] = \boxed{-2\pi i}$

$\therefore I = I_1 - I_2 = 2\pi i - (-2\pi i) = \boxed{4\pi i}$

3-c) Expand $f(z) = \frac{\sin z}{(z-\pi)^2}$ in Laurent's series near $z = \pi$ and find the residue at this point

Taylor series $\rightarrow \sin z = \sin \pi + (z-\pi)\cos \pi - \frac{1}{2!}(z-\pi)^2 \sin \pi - \frac{1}{3!}(z-\pi)^3 \cos \pi + \cdots$

$\therefore \sin z = -(z-\pi) - \frac{1}{3!}(z-\pi)^3 - \frac{1}{5!}(z-\pi)^5 + \cdots$

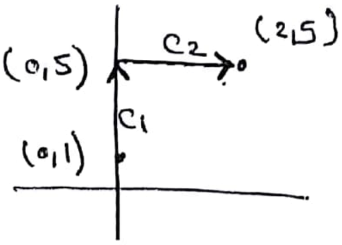
$$f(z) = \frac{1}{(z-\pi)^2}\left[-(z-\pi) - \frac{1}{3!}(z-\pi)^3 - \frac{1}{5!}(z-\pi)^5 + \cdots\right]$$

$$\boxed{f(z) = \frac{-1}{(z-\pi)} - \frac{1}{3!}(z-\pi) - \frac{1}{5!}(z-\pi)^3 - \cdots}$$

$$\boxed{\operatorname{Res}_{z=\pi} f(z) = -1}$$

-d- Evaluate $\int_{(0,1)}^{(2,5)} (3x+y)\,dx + (2y-x)\,dy$ along the straight line joining from $(0,1)$ to $(0,5)$ and from $(0,5)$ to $(2,5)$.

on C_1: $x=0 \longrightarrow dx=0$



$$I_{C_1} = \int_{y=1}^{y=5} (2y-0)\,dy = \left[y^2\right]_1^5 = 25-1 = 24$$

on C_2: $y=5 \longrightarrow dy=0$

$$I_{C_2} = \int_{x=0}^{x=2} (3x+5)\,dx = \left(\frac{3x^2}{2}+5x\right)\Big|_0^2 = \left(\frac{3(4)}{2}+10\right) - 0 = 16$$

$$I = I_{C_1} + I_{C_2} = 24 + 16 = \boxed{40}$$