نموذج اجابة

المادة : رياضيات هندسية ٢-ب

Engineering Mathematics 2-B

PHM1006

الفرقة : الأولى كهرباء

الفصل الدراسي الثاني

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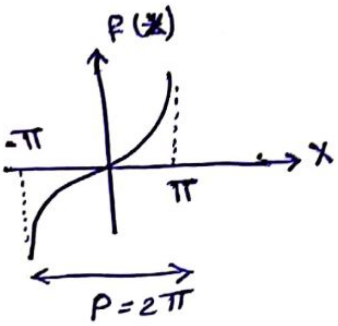
1 a) Determine the Fourier sine and Cosine series of the function: $f(x) = x^2 \quad 0 < x < \pi$

① sine series



* $f(x)$ is periodic function $p = 2\pi$, $L = \pi$
* $f(x)$ is picewise continous function
* odd function
* not harmonic function

$a_0 = a_n = 0 \qquad b_n = ✓$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

u	dv
x^2	$\sin nx$
$2x$ (+)	$\frac{-\cos nx}{n}$
2 (−)	$\frac{-\sin nx}{n^2}$
0 (+)	$\frac{\cos nx}{n^3}$

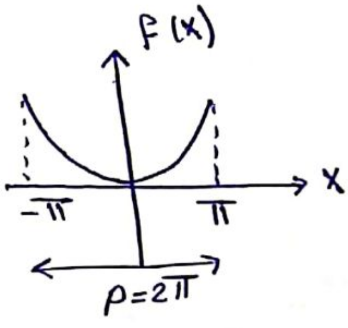
$$b_n = \frac{2}{\pi}\left[-x^2\frac{\cos nx}{n} + 2x\frac{\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$

$$b_n = \frac{-2\pi(-1)^n}{n} + \frac{4[(-1)^n - 1]}{n^3}$$

$$f(x) = x^2 = \sum_{n=1}^{\infty}\left[\frac{-2\pi(-1)^n}{n} + \frac{4((-1)^n - 1)}{n^3}\right]\sin nx$$

② Cosine series



* $f(x)$ is periodic function $p = 2\pi$
* $f(x)$ is continous function
* even function
* not harmonic function

$b_n = 0 \qquad a_0 = ✓ \qquad a_n = ✓$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}\left[x^3\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[x^2\,\frac{\sin nx}{n} + 2x\,\frac{\cos nx}{n^2} - 2\,\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

u		dv
x^2		$\cos nx$
$2x$	+	$\frac{\sin nx}{n}$
2	−	$-\frac{\cos nx}{n^2}$
0	+	$-\frac{\sin nx}{n^3}$

b) Expand $f(t)=t^2$ on $[-\pi,\pi]$ in a complex Fourier series.

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{-im\pi x}{L}}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, e^{-imt}\, dt$$

$$= \frac{1}{2\pi}\left[t^2\,\frac{e^{-imt}}{-im} + \frac{2t}{m^2}\,e^{-imt} + \frac{2}{im^3}\,e^{-imt}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{-im}\,e^{-im\pi} + \frac{2\pi}{m^2}\,e^{-im\pi} + \frac{2}{im^3}\,e^{-im\pi} + \frac{\pi^2}{im}\,e^{im\pi} + \frac{2\pi}{m^3}\,e^{im\pi} - \frac{2}{im^3}\,e^{im\pi}\right]$$

u		dv
t^2		e^{-imt}
$2t$	+	$e^{-imt}/-im$
2	−	$e^{-imt}/-m^2$
0	+	e^{-imt}/im^3

where $e^{i\theta} = \cos\theta + i\sin\theta \longrightarrow e^{-i\theta} = \cos\theta - i\sin\theta$

$$e^{-im\pi} = \cos m\pi - i\,\underbrace{\sin m\pi}_{\to 0} = (-1)^m \quad , \quad e^{im\pi} = (-1)^m$$

$$c_m = \frac{1}{2\pi}(-1)^m\left[\cancel{\frac{\pi^2}{-im}} + \frac{2\pi}{m^2} + \cancel{\frac{2}{im^3}} + \cancel{\frac{\pi^2}{im}} + \frac{2\pi}{m^2} - \cancel{\frac{2}{im^3}}\right]$$

$$\boxed{c_m} = \frac{(-1)^m\, 4\cancel{\pi}}{2\cancel{\pi}\, m^2} = \boxed{\frac{2(-1)^m}{m^2}} \qquad m \neq 0$$

$$\boxed{c_0} = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt = \frac{1}{6\pi}\left[x^3\right]_{-\pi}^{\pi} = \boxed{\frac{\pi^2}{3}}$$

$$f(x) = c_0 + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} c_m\, e^{\frac{im\pi t}{L}}$$

$$\boxed{f(x) = t^2 = \frac{\pi^2}{3} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}\frac{2(-1)^m}{m^2}\, e^{imt}}$$

c) show that the set $\{\cos(nx)\}, n \in N$ is orthogonal but not orthonormal on $[0, 2\pi]$

$\{\cos x, \cos 2x, \cos 3x, \ldots\}$

① $\int_0^{2\pi} \cos mx \ \cos nx \ dx \qquad m \neq n$

$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x] dx$

$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi}$

$= \frac{1}{2}\left[\frac{\sin 2(m+n)\pi}{m+n} + \frac{\sin 2(m-n)\pi}{m-n}\right] = \text{Zero}$

② $\int_0^{2\pi} \cos nx \ \cos nx \ dx = \int_0^{2\pi} \cos^2 nx \ dx$

$= \frac{1}{2}\int_0^{2\pi} (1+\cos 2nx)\,dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \pi \neq 1$

$\therefore$ set $\{\cos nx\}$ is orthogonal but not orthonormal.

d) A refinery produces both gasoline and fule oil ......

Let X : is the number of gallons of gasoline produced

y : is " " " " fule oil "

① objective function $f = 1X + 0.9y$

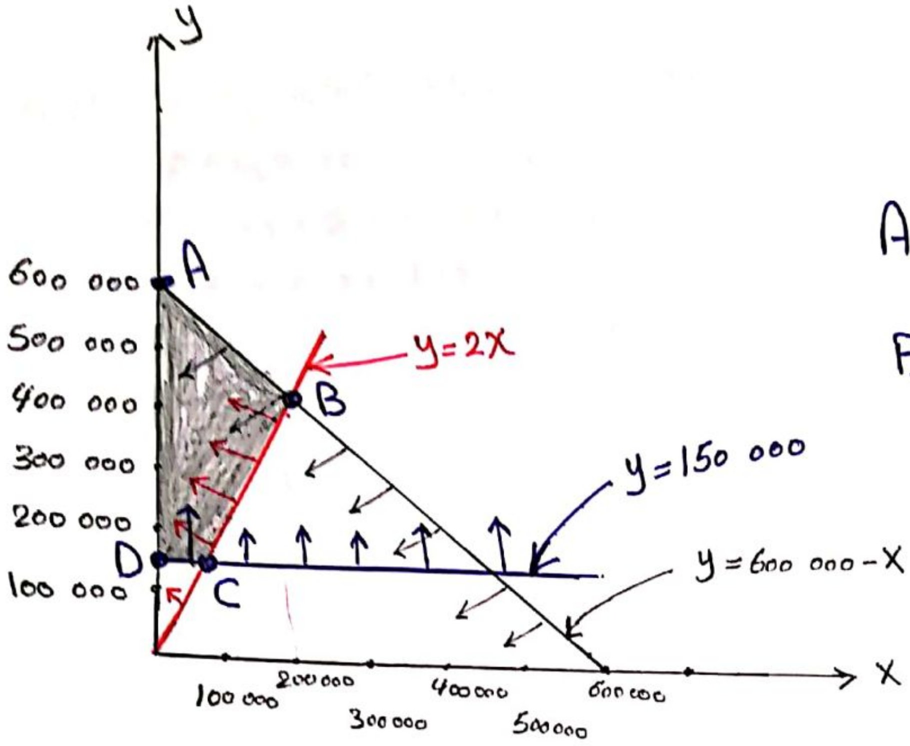
② constranits : $X \geq 0$, $y \geq 0$

$x + y \leq 600\,000 \longrightarrow y = 600\,000 - X$

$y \geq 2X \longrightarrow y = 2X$

$y \geq 150000 \longrightarrow y = 150\,000$

③ feasible region



A (0, 600 000)

B → y = 2X, y = 600 000 − X

B = (200 000, 400 000)

C → y = 150 000, y = 2X

C = (75000, 150 000)

D = (0, 150 000)

$F|_{A(0,600\,000)} = X + 0.9Y = 0.9(600\,000) = \$\,540\,000$

$F|_{B(200\,000,\,400\,000)} = 200\,000 + 0.9(400\,000) = \$\,560\,000$

$F|_{C(75\,000,\,150\,000)} = 75\,000 + 0.9(150\,000) = \$210\,000$

$F|_{D(0,150\,000)} = 0.9(150\,000) = \$135\,000$

max profit is $ 560 000 at X = 200 000 & y = 400 000

[2] a) Using MSV method to solve the heat equation:

$8u_{xx} = u_t \qquad 0<x<5 \qquad u(0,t)=0 \qquad u(5,t)=0$

$u(x,0) = 2\sin(\pi x) - 4\sin(2\pi x) + \sin(5\pi x)$

$u(x,t) = X(x)\cdot T(t)$

$u_x = X' T \qquad u_t = X\dot{T}$

$u_{xx} = X'' T$

$X'' T = \frac{1}{8} X\dot{T} \qquad \div XT$

$\frac{X'' T}{XT} = \frac{1}{8}\frac{X\dot{T}}{XT}$

$\frac{X''}{X} = \frac{1}{8}\frac{\dot{T}}{T} = \lambda = \begin{cases} 0 \\ <0 \\ >0 \end{cases}$

case 3 $\quad \lambda<0 \implies \lambda = -p^2$

$\frac{X''}{X} = -p^2 \qquad\qquad \frac{1}{8}\frac{\dot{T}}{T} = -p^2$

$X'' = -p^2 X \to X'' + p^2 X = 0 \qquad\qquad \dot{T} + 8p^2 T = 0$

$(\alpha^2 + p^2)X = 0 \qquad\qquad (D + 8p^2)T$

$\alpha^2 = -p^2 \qquad\qquad \alpha = -8p^2$ real not repeated

$\alpha_{1,2} = \pm ip$ complex not repeated $\qquad \boxed{T(t) = C_3 e^{-8p^2 t}}$

$\boxed{X(x) = C_1\cos px + C_2\sin px}$

B.C 1 $\quad u(0,t)=0 \qquad X(0)\,T(t) = 0$

$\therefore X(0) = 0 \to \boxed{C_1 = 0}$

B.C 2 $\quad u(5,t) = 0 \qquad X(5)\,T(t) = 0$

$\therefore X(5) = 0 \to C_2 \sin 5p = 0$

$C_2 \neq 0 \to \sin 5p = 0 \qquad 5p = n\pi \qquad \boxed{p = \frac{n\pi}{5}}$

$\boxed{\therefore X(x) = C_2 \sin\frac{n\pi x}{5}}$

$u(x,t) = C_2 C_3 \sin\frac{n\pi x}{5}\, e^{-8\left(\frac{n^2\pi^2}{25}t\right)}$

$= \sum C_n \sin\left(\frac{n\pi x}{5}\right) e^{-8\left(\frac{n^2\pi^2}{25}t\right)}$

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I.C ① $u(x,0) = 2\sin(\pi x) - 4\sin(2\pi x) + \sin(5\pi x)$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{5}\right)$$

$n=5$, $n=10$, $n=25$

$$c_5 \sin(\pi x) + c_{10}\sin(2\pi x) + c_{25}\sin(5\pi x) + \cdots\cdots$$

$$= 2\sin(\pi x) - 4\sin(2\pi x) + \sin(5\pi x)$$

$\therefore$ $\boxed{c_5 = 2}$ $\boxed{c_{10} = -4}$ $\boxed{c_{25} = 1}$

$$\therefore u(x,t) = 2\sin(\pi x)\, e^{-8\left(\frac{5^2\pi^2 t}{25}\right)} - 4\sin(2\pi x)\, e^{-8\left(\frac{10^2\pi^2 t}{25}\right)} + \sin(5\pi x)\, e^{-8\left(\frac{25^2\pi^2 t}{25}\right)}$$

$$\boxed{\therefore u(x,t) = 2e^{-8\pi^2 t}\sin(\pi x) - 4e^{-32\pi^2 t}\sin(2\pi x) + e^{-200\pi^2 t}\sin(5\pi x)}$$

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b) Use D'Alembert's method to solve $u_{tt} = \frac{1}{\pi^2}u_{xx}$, $t \geq 0$
$u(x,0) = \cos(\pi x)$, $u_t(x,0) = 0$, $-\infty < x < \infty$

$$u_{xx} = \pi^2 u_{tt} \Longrightarrow u_{xx} = \frac{1}{c^2}u_{tt} \Longrightarrow \frac{1}{c^2} = \pi^2$$

$$c^2 = \frac{1}{\pi^2} \Longrightarrow \boxed{c = \frac{1}{\pi}}$$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$

$$= \frac{1}{2}\left[\cos\pi\left(x+\frac{t}{\pi}\right) + \cos\pi\left(x-\frac{t}{\pi}\right)\right] + \frac{1}{2\pi}\cancel{\int 0\,dy}^{\,0}$$

$$= \frac{1}{2}\left[\cos(\pi x + t) + \cos(\pi x - t)\right]$$

$$\boxed{\therefore u(x,t) = \cos\pi x \cdot \cos t}$$

c) solve the p.D.E by Euler solution:

$16u_{xx} - 8u_{xy} + u_{yy} = 0$ find particular solution which satisfies $u(x,0) = e^{3x}$, $u(0,y) = 4y^2 + e^{12y}$

Let $\lambda^2 = u_{yy}$ $\quad \lambda = u_{xy}$ $\quad 1 = u_{xx}$

$$\lambda^2 - 8\lambda + 16 = 0 \longrightarrow (\lambda-4)(\lambda-4) = 0$$

$$\boxed{\lambda = \lambda_1 = \lambda_2 = 4}$$

$$u(x,y) = F(x+\lambda y) + y\,G(x+\lambda y)$$

$$u(x,y) = F(x+4y) + y\,G(x+4y)$$

$$u(x,0) = e^{3x} = f(x)$$

$$\therefore f(x) = e^{3x}$$

$$f(x+4y) = e^{3(x+4y)} = e^{3x+12y}$$

$$u(x,y) = e^{3x+12y} + y\,G(x+4y)$$

$$\therefore u(0,y) = 4y^2 + \cancel{e^{12y}} = \cancel{e^{12y}} + y\,G(4y)$$

$$4y^{\cancel{2}} = \cancel{y}\,G(4y) \Longrightarrow 4y = G(4y)$$

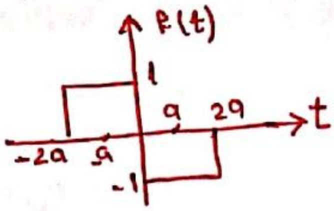
$$\therefore G(x+4y) = x+4y$$

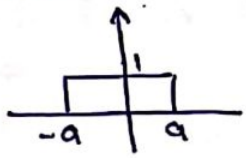
$$\therefore u(x,y) = e^{3x+12y} + y(x+4y)$$

$$\boxed{u(x,y) = e^{3x+12y} + xy + 4y^2}$$

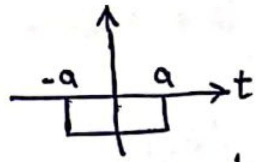
3 a) Find Fourier transform of

i)





$f(t) = Rec\left(\frac{t+a}{2a}\right)$



$f(t) = -Rec\left(\frac{t-a}{2a}\right)$

$$f(t) = Rec\left(\frac{t+a}{2a}\right) - Rec\left(\frac{t-a}{2a}\right)$$

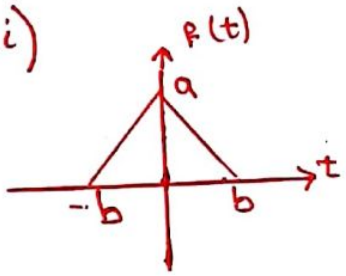
$$F[f(t)] = \hat{f}(w) = e^{iaw}\, F\left[Rec\left(\frac{t}{2a}\right)\right] - e^{-iaw}\, F\left[Rec\left(\frac{t}{2a}\right)\right]$$

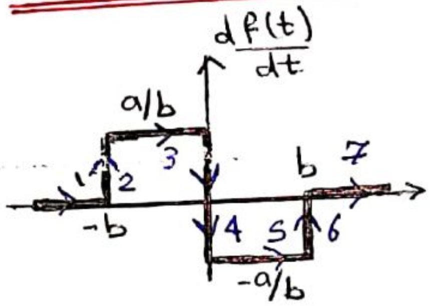
$$\because f(t) = A\, rec\left(\frac{t}{T}\right) \Longrightarrow \hat{f}(\omega) = AT\, sinc\left(\frac{\omega T}{2}\right)$$

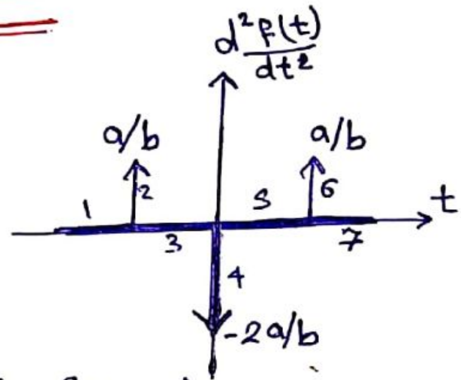
$$\hat{f}(w) = \left(\frac{e^{iaw} - e^{-iaw}}{2i}\right) 2i \;\; 2a\, sinc\left(\frac{w\cancel{2}a}{\cancel{2}}\right)$$

$$\boxed{\hat{f}(\omega) = 4ia\, \sin(wa)\, sinc(wa)}$$

ii)







$$\frac{d^2f(t)}{dt^2} = a/b\, \delta(t+b) - 2a/b\, \delta(t) + a/b\, \delta(t-b)$$

$$\frac{d^2f(t)}{dt^2} = a/b\, [\delta(t+b) - 2\delta(t) + \delta(t-b)]$$

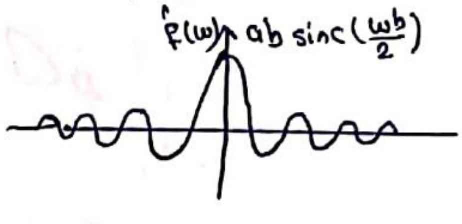
$$(i\omega)^2\, \hat{f}(\omega) = a/b\left[e^{i\omega b} - 2 + e^{-i\omega b}\right]$$

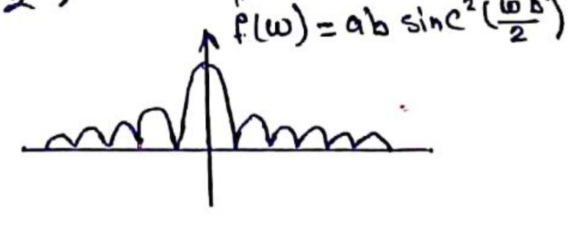
$$-\omega^2\, \hat{f}(\omega) = a/b\left[2\left(\frac{e^{i\omega} + e^{-i\omega b}}{2}\right) - 2\right]$$

$$\hat{f}(\omega) = \frac{-2a}{\omega^2 b}[\cos\omega b - 1] = \frac{2a}{\omega^2 b}[1 - \cos\omega b]$$

$$\hat{f}(\omega) = \frac{2a}{\omega^2 b}\left[2\sin^2\left(\frac{\omega b}{2}\right)\right] = \frac{4a}{\omega^2 b}\left[\frac{\sin\left(\frac{\omega b}{2}\right)}{\left(\frac{\omega b}{2}\right)}\right]^2\left(\frac{\omega b}{2}\right)^2$$

$$\hat{f}(\omega) = ab\, sinc^2\left(\frac{\omega b}{2}\right)$$





b) Find Fourier transform of $f(t) = e^{-a^2t^2}$ where a is constant and $\int_0^\infty e^{-u^2}\, du = \frac{\sqrt{\pi}}{2}$

$$F[f(t)] = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-a^2t^2} e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} e^{-(a^2t^2 + i\omega t)}\, dt$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-\left[\left(at + \frac{i\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right]}\, dt$$

$$\hat{f}(\omega) = e^{\frac{-\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(at + \frac{i\omega}{2a}\right)^2}\, dt$$

Let $u = at + \frac{i\omega}{2a} \Longrightarrow du = a\, dt \Longrightarrow dt = \frac{du}{a} \Longrightarrow$ $t: -\infty \to \infty$, $u: -\infty \to \infty$

$$\hat{f}(\omega) = e^{\frac{-\omega^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a} = \frac{1}{a} e^{\frac{-\omega^2}{4a^2}} \underbrace{\int_{-\infty}^{\infty} e^{-u^2}\, du}_{\text{even function}}$$

where $\int_0^\infty e^{-u^2}\, du = \sqrt{\pi}/2 \Longrightarrow$ from gamma function

$$\hat{f}(\omega) = \frac{1}{a} e^{\frac{-\omega^2}{4a^2}} \cdot 2\int_0^\infty e^{-u^2}\, du = \frac{1}{a} e^{\frac{-\omega^2}{4a^2}}\, 2\frac{\sqrt{\pi}}{2}$$

$$\boxed{\hat{f}(\omega) = \frac{\sqrt{\pi}}{a} e^{\frac{-\omega^2}{4a^2}}}$$

c) Find the directional derivative of f at p in the direction of $\vec{a}$ in the following:

$f(x, y, z) = xyz$, $p(-1, 1, 3)$, $\vec{a} = [1, -2, 2]$

$|\bar{a}| = \sqrt{1+4+4} = \sqrt{9} = 3 \neq 1$ is not unit vector

$$D_{\bar{a}} f = \frac{\bar{a}}{|\bar{a}|} \cdot \bar{\nabla} f \Longrightarrow \bar{\nabla} f = \text{grad } f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$$

$$\bar{\nabla} f = yz\,\underline{i} + xz\,\underline{j} + xy\,\underline{k}$$

$$D_{\bar{a}} f = \frac{[1, -2, 2]}{3} (yz\,\underline{i} + xz\,\underline{j} + xy\,\underline{k}) = \frac{yz - 2xz + 2xy}{3}$$

$$D_{\bar{a}} f\Big|_{p(-1,1,3)} = 3 + 6 - 2 = 7/3$$

d) Compute the curl of the following vector field $\vec{F}(x,y,z) = [e^y \cos x, e^y \sin x, 0]$ and show that vector $\vec{F}$ is conservative or not

$$\vec{F}(x,y,z) = [e^y \cos x, e^y \sin x, 0]$$

$$\operatorname{curl} \vec{F} = \overline{\nabla} \times \overline{F} = \begin{vmatrix} \overset{+}{i} & \overset{-}{J} & \overset{+}{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y \cos x & e^y \sin x & 0 \end{vmatrix}$$

$$\operatorname{curl} \vec{F} = (e^y \cos x - e^y \cos x) \underline{k} = [0, 0, 0] = 0$$

$\therefore \vec{F}$ is conservative

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