Kafrelsheikh University
Faculty of Engineering
Dept. of Mech. Engineering
4th year – Electrical Eng.
Subject: Robotic Systems

1st Semester
Final Exam
Date: Dec. 25th, 2017
Time allowed: 3 hours
Full Mark: 70

Question 1 (24 Marks):

a. **(4 Marks) (ILO a.3.1 pass 11/21)** Mention two different methods that are used to control the robot arm.

Solution:

Examples of control methods:

Computed torque control.

Feedforward nonlinear control.

Resolved rate control.

b. **(10Marks) (ILO c.5.1 pass 14/21)** Design a trajectory-following controller for a system with dynamics given by

$$f = ax^2\dot{x}\ddot{x} + b\dot{x}^2 + c\sin(x)$$

such that errors are suppressed in a critically damped fashion over all configurations.

Solution:

$$f = \alpha \acute{f} + \beta$$

$$\alpha = ax^2\dot{x}$$

$$\beta = b\dot{x}^2 + c\sin(x)$$

$$\acute{f} = \ddot{x}_d - k_v\,\dot{e} - k_p\,e$$

Where $e = x_d - x$

Then the closed loop equation for the system is

$$ax^2\dot{x}\left(\ddot{x}_d - k_v\,\dot{e} - k_p\,e\right) + b\dot{x}^2 + c\sin(x) = ax^2\dot{x}\ddot{x} + b\dot{x}^2 + c\sin(x)$$

$$\ddot{e} - k_v\,\dot{e} - k_p\,e = 0$$

For $k_v = 2\sqrt{k_p}$, the error is suppressed in a critically damped fashion over all configurations.

c. **(10 Marks)** Consider a manipulator with dynamics given by

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

and is to be controlled such that (without loss of generality) attempts to maintain $\Theta_d = 0$. Prove that the control law

$$\tau = -K_p\Theta - M(\Theta)K_v\dot{\Theta} + G(\Theta)$$

yields an asymptotically stable nonlinear system. You may take K_v to be of the form $K_v = k_v I_n$ where k_v is a scalar and I_n is the $n \times n$ identity matrix.

Solution:

$$M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) = -K_p\Theta - M(\Theta)K_v\dot{\Theta} + G(\Theta)$$

Choose Lyapunov candidate function as

$$V = \frac{1}{2}\,\dot{\Theta}^T M(\Theta)\dot{\Theta} + \frac{1}{2}\Theta^T K_p \Theta$$

Then we calculate the time derivative of the Lyapunov candidate function

$$\dot{V} = \frac{1}{2}\,\dot{\Theta}^T \dot{M}(\Theta)\dot{\Theta} + \dot{\Theta}^T M(\Theta)\ddot{\Theta} + \dot{\Theta}^T K_p \Theta$$

$$\dot{V} = \frac{1}{2}\,\dot{\Theta}^T \dot{M}(\Theta)\dot{\Theta} + \dot{\Theta}^T\left[-V(\Theta,\dot{\Theta}) - G(\Theta) - K_p\Theta - M(\Theta)K_v\dot{\Theta} + G(\Theta)\right] + \dot{\Theta}^T K_p\Theta$$

$$\dot{V} = \frac{1}{2}\,\dot{\Theta}^T \dot{M}(\Theta)\dot{\Theta} + \dot{\Theta}^T\left[-V(\Theta,\dot{\Theta}) - M(\Theta)K_v\dot{\Theta}\right]$$

$V(\Theta, \dot{\Theta})$ can be written in the form $V(\Theta, \dot{\Theta}) = C(\Theta, \dot{\Theta})\,\dot{\Theta}$, then

$$\dot{V} = \dot{\Theta}^T\left[\frac{1}{2}\dot{M}(\Theta) - C(\Theta,\dot{\Theta}) - M(\Theta)K_v\right]\dot{\Theta}$$

$$\dot{V} = \frac{1}{2}\,\dot{\Theta}^T\left[\dot{M}(\Theta) - 2C(\Theta,\dot{\Theta})\right]\dot{\Theta} - \dot{\Theta}^T M(\Theta)K_v\dot{\Theta}$$

From the dynamics of robot manipulators, $\dot{M}(\Theta) - 2C(\Theta,\dot{\Theta})$ is a skew symmetric matrix, then

$$\dot{\Theta}^T\left[\dot{M}(\Theta) - 2C(\Theta,\dot{\Theta})\right]\dot{\Theta} = 0$$

Then

$$\dot{V} = -\,\dot{\Theta}^T M(\Theta)K_v\dot{\Theta}$$

Since $M(\Theta)$ is a positive definite matrix and $K_v = k_v I_n$, k_v is a positive scalar, then

$M(\Theta)K_v$ is a positive definite matrix.

Then $\dot{V}$ is a negative definite and the system is asymptotically stable

Question 2 (10 Marks) (ILO b.1.2 pass 19/21):

A single-link robot with a rotary joint is motionless at $\theta = -10°$. It is desired to move the joint in a smooth manner to $\theta = 50°$ in 4 seconds. Find the coefficients of a cubic which accomplishes this motion and brings the arm to rest at the goal. Find the maximum velocity and maximum acceleration during the motion.

Solution:

The cubic is in the form:

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

where

$$a_0 = \theta_0, \quad a_1 = 0, \quad a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0), \quad a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

Then,

$$a_0 = -10, \quad a_1 = 0, \quad a_2 = \frac{3}{16}(50 + 10) = 11.25, \quad a_3 = -\frac{2}{64} * 60 = -1.875$$

$$\theta(t) = -10 + 11.25t^2 - 1.875\, t^3$$

$$\dot{\theta}(t) = 22.50\, t - 5.625t^2$$

$$\ddot{\theta}(t) = 22.50 - 11.25\, t$$

Max. velocity at $\ddot{\theta}(t) = 22.50 - 11.25\, t = 0$ at $t = 2$ s

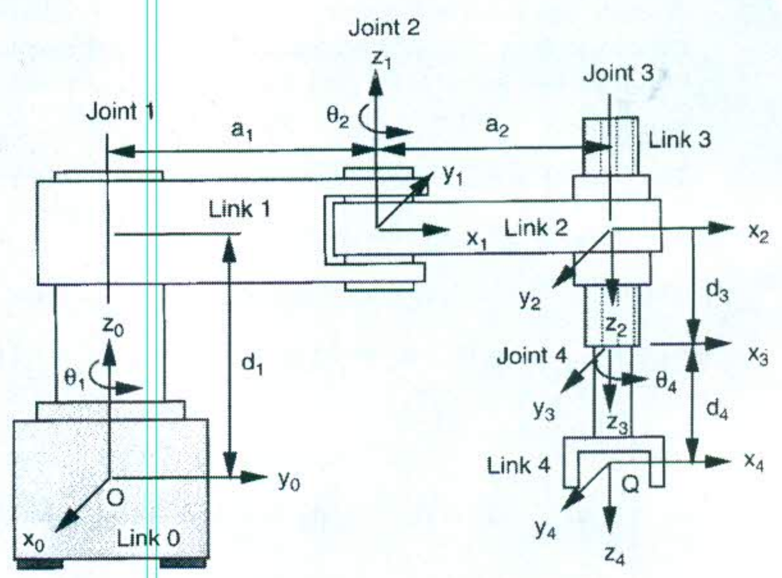
Max. velocity $\dot{\theta}(2) = 22.50 * 2 - 5.625 * 4 = 22.5$ deg/s.

Max. acceleration is at $t = 0$ or at $t = 4$ s

Max. acceleration = 22.5 deg/s^2

Question 3 (12 Marks) (ILO c.5.2 Pass 21/21):

For the SCARA arm shown in the figure, establish the D-H and overall transformation matrices ${}^{i-1}A_i$ for $i = 1, 2, 3, 4$ and ${}^{0}A_4$.



Joint i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	θ_1
2	π	a_2	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

D-H Parameters

$$ {}^{0}A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1 c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1 s\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}A_2 = \begin{bmatrix} c\theta_2 & s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & -c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

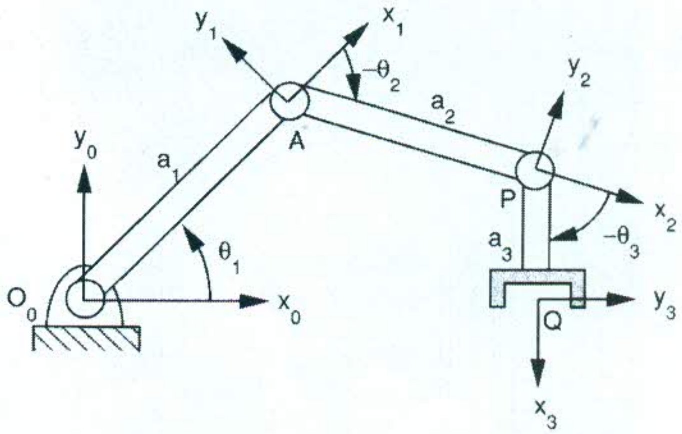
$$ {}^{3}A_4 = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & 0 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}A_2 = \begin{bmatrix} c_{12} & s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & -c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{0}A_3 = \begin{bmatrix} c_{12} & s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & -c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & d_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}A_4 = \begin{bmatrix} c_{12-4} & s_{12-4} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12-4} & -c_{12-4} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & d_1 - d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Question 4 (15 Marks) (ILO a.3.2 Pass 18/21):

A planar 3-dof, 3R manipulator shown in the figure. We wish to move the end effector, Q, along the x_0-axis at 1.0 m/s and at the same time, keep the direction of the approach, the x_3-axis, in the y_0-direction. Calculate the joint rates required to accomplish this task. Under what conditions will the joint rates approach infinity.



Solution:

$$^0A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$^0A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$^0A_3 = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$z_0 = z_1 = z_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ ^0P_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix},\ ^0P_2 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix},\ ^0P_3 = \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix}$$

$$^0P_3^* = {}^0P_3 - {}^0P_0 = \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix},\ ^1P_3^* = {}^0P_3 - {}^0P_1 = \begin{bmatrix} a_2c_{12} + a_3c_{123} \\ a_2s_{12} + a_3s_{123} \\ 0 \end{bmatrix}$$

$$^2P_3^* = {}^0P_3 - {}^0P_2 = \begin{bmatrix} a_3c_{123} \\ a_3s_{123} \\ 0 \end{bmatrix}$$

$$J_i = \begin{bmatrix} z_{i-1} \times {}^{i-1}P_3^* \\ z_{i-1} \end{bmatrix}$$ all the joints are revolute.

$$J_1 = \begin{bmatrix} z_0 \times {}^0P_3^* \\ z_0 \end{bmatrix} = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},\ J_2 = \begin{bmatrix} z_1 \times {}^1P_3^* \\ z_1 \end{bmatrix} = \begin{bmatrix} -a_2s_{12} - a_3s_{123} \\ a_2c_{12} + a_3c_{123} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} z_2 \times {}^2P_3^* \\ z_2 \end{bmatrix} = \begin{bmatrix} -a_3s_{123} \\ a_3c_{123} \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$ then

$$J = [J_1 \quad J_2 \quad J_3] = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

$$|J| = a_1a_2s_2$$

$$J^{-1} = \frac{1}{a_1a_2s_2}\begin{bmatrix} a_2\,c_{12} & a_2\,s_{12} & a_2a_3s_3 \\ -a_1c_1 - a_2c_{12} & -a_1s_1 - a_2s_{12} & -a_3a_2s_2 - a_1a_3s_{23} \\ a_1\,c_1 & a_1\,s_1 & a_1a_2s_2 + a_1a_3s_{23} \end{bmatrix}$$

$\dot{Z} = J\dot{\theta}$ $\qquad\qquad$ $\dot{\theta} = J^{-1}\dot{Z},\ \dot{Z} = \begin{bmatrix} 1.0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{a_1a_2s_2}\begin{bmatrix} a_2\,c_{12} & a_2\,s_{12} & a_2a_3s_3 \\ -a_1c_1 - a_2c_{12} & -a_1s_1 - a_2s_{12} & -a_3a_2s_2 - a_1a_3s_{23} \\ a_1\,c_1 & a_1\,s_1 & a_1a_2s_2 + a_1a_3s_{23} \end{bmatrix}\begin{bmatrix} 1.0 \\ 0 \\ 0 \end{bmatrix}$$
$$= \frac{1}{a_1a_2s_2}\begin{bmatrix} a_2\,c_{12} \\ -a_1c_1 - a_2c_{12} \\ a_1\,c_1 \end{bmatrix}$$

$$\dot{\theta}_1 = \frac{a_2\,c_{12}}{a_1a_2s_2} = \frac{c_{12}}{a_1s_2}$$

$$\dot{\theta}_2 = \frac{-a_1c_1 - a_2c_{12}}{a_1a_2s_2}$$

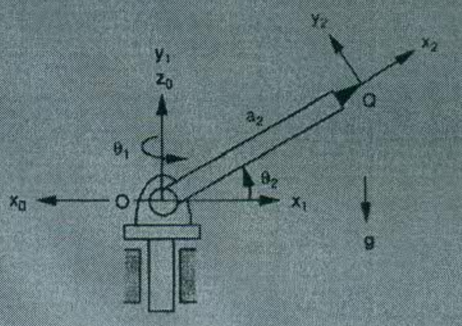
$$\dot{\theta}_3 = \frac{a_1\,c_1}{a_1a_2s_2} = \frac{c_1}{a_2s_2}$$

The joint rates will approach infinity when $s_2 = 0$ i.e. when $\theta_2 = 0\ or\ \pi$.

ILO b.1.1 (pass 6/12)

Question 5 (15 Marks):

Consider the 2-dof pointer shown in the figure. Assuming that the inertia of the first link is negligible and that the second link is a slender homogeneous rod of mass m, drive the dynamical equations of motion by the Lagrangian method using θ_1 and θ_2 as the generalized coordinates.



Sol. :- $^1I_1 = [0]$, $^2I_2 = \frac{1}{12} m_2 a_2^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I_1 = {}^0R_1 \, {}^1I_1 \, {}^0R_1^T = [0]$$

$$I_2 = {}^0R_2 \, {}^2I_2 \, {}^0R_2^T$$

$$= \frac{1}{12} m_2 a_2^2 \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 \\ C_2S_1 & -S_1S_2 & -C_1 \\ S_2 & C_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^0R_2^T$$

$$= \frac{1}{12} m_2 a_2^2 \begin{bmatrix} 0 & -C_1S_2 & S_1 \\ 0 & -S_1S_2 & -C_1 \\ 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} C_1C_2 & C_2S_1 & S_2 \\ -C_1S_2 & -S_2S_1 & C_2 \\ S_1 & -C_1 & 0 \end{bmatrix}$$

$$= \frac{1}{12} m_2 a_2^2 \begin{bmatrix} C_1^2S_2^2 + S_1^2 & C_1S_1S_2^2 - S_1C_1 & -C_1C_2S_2 \\ C_1S_1S_2^2 - C_1S_1 & C_1^2 + S_1^2S_2^2 & -S_1S_2C_2 \\ -C_1C_2S_2 & -S_2C_2S_1 & C_2^2 \end{bmatrix}$$

Joint	α_i	a_i	d_i	θ_i
1	$\frac{\pi}{2}$	0	0	θ_1
2	0	a_2	0	θ_2

$$^0A_1 = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1A_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2C_2 \\ S_2 & C_2 & 0 & a_2S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0A_2 = \begin{bmatrix} C_1C_2 & -C_1S_2 & S_1 & a_2 \\ C_2S_1 & -S_1S_2 & -C_1 & a_2 \\ S_2 & C_2 & 0 & a \\ 0 & 0 & 0 & \end{bmatrix}$$

Link Jacobian matrix :-

Link 1 :-

$$J_{v_1}^1 = Z_0 \times {}^0P_{c_1}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \quad Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{\omega_1}^1 = Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{v_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} , \quad J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Link 2 :-

$$J_{v_2}^1 = Z_0 \times {}^0P_{c_2}^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2}a_2C_1C_2 \\ \frac{1}{2}a_2S_1C_2 \\ \frac{1}{2}a_2S_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a_2S_1C_2 \\ \frac{1}{2}a_2C_1C_2 \\ 0 \end{bmatrix}$$

$$J_{v_2}^2 = Z_1 \times {}^1P_{c_2}^* = \begin{bmatrix} S_1 \\ -C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{2}a_2C_1C_2 \\ \frac{1}{2}a_2S_1C_2 \\ \frac{1}{2}a_2S_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a_2C_1S_2 \\ -\frac{1}{2}a_2S_1S_2 \\ \frac{1}{2}a_2S_1^2C_2 + \frac{1}{2}a_2C_1^2C_2 \end{bmatrix}$$

Question # 5 Solution (Continued)

Manipulator Inertia matrix

$$M = \sum_{i=1}^{2} \left(J_{v_i}^T m_i J_{v_i} + J_{w_i}^T I_i J_{w_i} \right)$$

$$= J_{v_1}^T m_1 J_{v_1} + J_{w_1}^T I_1 J_{w_1} + J_{v_2}^T m_2 J_{v_2} + J_{w_2}^T I_2 J_{w_2}$$

$$J_{v_2}^T m_2 J_{v_2} = m_2 \begin{bmatrix} -\frac{1}{2} a_2 s_1 c_2 & \frac{1}{2} a_2 c_1 c_2 & 0 \\ -\frac{1}{2} a_2 c_1 s_2 & -\frac{1}{2} a_2 s_1 s_2 & \frac{1}{2} a_2 c_2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} a_2 s_1 c_2 & -\frac{1}{2} a_2 c_1 s_2 \\ \frac{1}{2} a_2 c_1 c_2 & -\frac{1}{2} a_2 s_1 s_2 \\ 0 & \frac{1}{2} a_2 c_2 \end{bmatrix}$$

$$\frac{1}{4} a_2^2 \left[c_1 s_2 s_1 c_2 - s_1 s_2 c_1 c_2 \right], \quad \frac{1}{4} a_2^2 \left[c_1^2 s_2^2 + s_1^2 s_2^2 \right] + \frac{1}{4} a_2^2 c_2^2 = \frac{1}{4} a_2^2$$

$$= m_2 \begin{bmatrix} \frac{1}{4} a_2^2 c_2^2 & 0 \\ 0 & \frac{1}{4} a_2^2 \end{bmatrix} = \frac{1}{4} m_2 a_2^2 \begin{bmatrix} c_2^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_{w_2}^T I_2 J_{w_2} = \frac{1}{12} m_2 a_2^2 \begin{bmatrix} 0 & 0 & 1 \\ s_1 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} c_1^2 s_2^2 + s_1^2 & c_1 s_1 s_2^2 - s_1 c_1 & -c_1 c_2 s_2 \\ c_1 s_1 s_2^2 - c_1 s_1 & c_1^2 + s_1^2 s_2^2 & -s_1 s_2 c_2 \\ -c_1 c_2 s_2 & -s_2 c_2 s_1 & c_2^2 \end{bmatrix} \begin{bmatrix} 0 & s_1 \\ 0 & -c_1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{1}{12} m_2 a_2^2 \begin{bmatrix} c_2^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = \frac{1}{4} m_2 a_2^2 \begin{bmatrix} c_2^2 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{12} m_2 a_2^2 \begin{bmatrix} c_2^2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = \frac{1}{3} m_2 a_2^2 \begin{bmatrix} c_2^2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} m_2 a_2^2 c_2^2 & 0 \\ 0 & \frac{1}{3} m_2 a_2^2 \end{bmatrix}$$

$$V_1 = \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial M_{1j}}{\partial \theta_k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial \theta_1} \right) \dot{\theta}_j \dot{\theta}_k$$

$$= \left(\frac{\partial M_{11}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_1} \right) \dot{\theta}_1 \dot{\theta}_1 + \left(\frac{\partial M_{11}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{12}}{\partial \theta_1} \right) \dot{\theta}_1 \dot{\theta}_2$$
$$+ \left(\frac{\partial M_{12}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{21}}{\partial \theta_1} \right) \dot{\theta}_2 \dot{\theta}_1 + \left(\frac{\partial M_{12}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_1} \right) \dot{\theta}_2 \dot{\theta}_2$$

$$V_1 = -\frac{2}{3} m_2 a_2^2 s_2 c_2 \dot{\theta}_1 \dot{\theta}_2$$

$$V_2 = \left(\frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_2} \right) \dot{\theta}_1^2 + \left(\frac{\partial M_{21}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{12}}{\partial \theta_2} \right) \dot{\theta}_1 \dot{\theta}_2$$
$$+ \left(\frac{\partial M_{22}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{21}}{\partial \theta_2} \right) \dot{\theta}_1 \dot{\theta}_2 + \left(\frac{\partial M_{22}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_2} \right) \dot{\theta}_2^2$$

$$V_2 = \frac{1}{3} m_2 a_2^2 s_2 c_2 \dot{\theta}_1^2 - \frac{2}{3} m_2 a_2^2 s_2 c_2 \dot{\theta}_1 \dot{\theta}_2$$

(5-2)

Question #5 (Continued)

$$G_i = -\sum_{j=1}^{2} m_j\, g^T J_{v_j}^{i} \qquad , \quad g = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix}$$

$$G_1 = -m_1 g^T J_{v_1}^{1} - m_2 g^T J_{v_2}^{1} \qquad , \quad J_{v_1}^{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= -m_2 * \begin{bmatrix} 0 & 0 & -9.81 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} a_2 S_1 C_2 \\ -\frac{1}{2} a_2 C_1 C_2 \\ 0 \end{bmatrix} = 0$$

$$G_2 = -m_2 g^T J_{v_1}^{2} - m_2 g J_{v_2}^{2} \qquad J_{v_2}^{2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= -m_2 \begin{bmatrix} 0 & 0 & -g \end{bmatrix} \begin{bmatrix} -\frac{1}{2} a_2 C_1 S_2 \\ -\frac{1}{2} a_2 S_1 S_2 \\ \frac{1}{2} a_2 C_2 \end{bmatrix}$$

$$G_2 = \frac{1}{2} m g * a_2 C_2$$

$$\sum_{J=1}^{n} M_{ij}\, \ddot{q}_j + V_i + G_i = Q_i \qquad , \quad M = \begin{bmatrix} \frac{1}{3} m_1 a_2^2 C_2^2 & 0 \\ 0 & \frac{1}{3} m_2 a_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$Q_1 = \frac{1}{3} m_2 a_2^2 C_2^2 \ddot{\theta}_1 - \frac{2}{3} m_2 a_2^2 S_2 C_2 \dot{\theta}_1 \dot{\theta}_2$$

$$Q_2 = \frac{1}{3} m_2 a_2^2 \ddot{\theta}_2 + \frac{1}{3} m_2 a_2^2 S_2 C_2 \dot{\theta}_1^2 - \frac{2}{3} m_2 a_2^2 S_2 C_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 g a_2 C_2$$