Q1:-

a) $f(z) = u(x,y) + iv(x,y)$

$\because f(z)$ is an analytic function.

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$

$\therefore u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

$\therefore u$ is a harmonic function and we can prove that $v(x,y)$ is a harmonic function also.

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b) (i) $f(z) = \sin x \cosh y + i \cos x \sinh y$

$u(x,y) = \sin x \cosh y$ $\qquad$ $v(x,y) = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\qquad$ $v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\qquad$ $v_y = \cos x \cosh y$

$\Rightarrow u_x = v_y$ and $u_y = v_x$

u, v, u_x, u_y, v_x and v_y are continuous for all values of (x,y), Then $f(z)$ is an entire function.

(ii) $f(z) = \dfrac{x - iy}{x^2 + y^2}$

$u = \dfrac{x}{x^2+y^2}$, $v = \dfrac{-y}{x^2+y^2}$

$u_x = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$, $v_x = \dfrac{2xy}{(x^2+y^2)^2}$

$u_y = \dfrac{-2xy}{(x^2+y^2)^2}$ $\qquad$ $v_y = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$

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$u_x = v_y$. and $u_y = -v_x$

$\therefore f(z)$ is an analytic function for all $(x, y) \neq (0, 0)$.

(iii) $f(z) = z\bar{\bar{z}} = (x+iy)(x-iy)$

$= x^2 + y^2$

$\therefore u(x,y) = x^2 + y^2$ $\quad v = 0$

$u_x = 2x$ $\quad v_x = v_y = 0$

$u_y = 2y$

$\therefore f(z)$ is not ~~an~~ analytic function expect at $(x, y) = (0, 0)$.

c) $u(x,y) = x e^x \cos y - y e^x \sin y$.

$u_x = x e^x \cos y + e^x \cos y - y e^x \sin y$

$u_y = -x e^x \sin y - y e^x \cos y - e^x \sin y$

$u_{xx} = x e^x \cos y + 2 e^x \cos y - y e^x \sin y$

$u_{yy} = -x e^x \cos y + y e^x \sin y - 2 e^x \cos y$

$\therefore u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic function.

$f'(z) = u_x - i u_y = x e^x \cos y + e^x \cos y - y e^x \sin y + i(x e^x \sin y + y e^x \cos y + e^x \sin y)$

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$\therefore f'(z) = z e^{z} + e^{z}$

$\therefore f(z) = z e^{z} + ic$ (by integration w.r.t z)

$f(z) = (x+iy)\, e^{x+iy} + ic = e^{x}(x+iy)(\cos y + i\sin y) + ic$

$\therefore V(x, y) = e^{x}(y\cos y + x\sin y) + c$ $\cancel{\cancel{}}$

Question 2 :-

a) $f(z) = u + iv$

$x = \frac{1}{2}(z+\bar{z})$, $y = \frac{1}{2i}(z-\bar{z})$

$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}$ $\frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i} = \frac{i}{2}$

$$\frac{\partial w}{\partial \bar{z}} = \frac{\partial(u+iv)}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}(u+iv)\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}(u+iv)\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{1}{2}[u_x + iv_x] + \frac{i}{2}[u_y + iv_y]$$

$$= \frac{1}{2}[u_x - v_y] + \frac{i}{2}[v_x + u_y]$$

we have for analytic function

$u_x = v_y$ and $v_x = -u_y$

$\therefore \frac{\partial w}{\partial \bar{z}} = 0$ $\cancel{\cancel{}}$

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b) $w = \sin^{-1} i$

$\therefore i = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$

$\therefore e^{iw} - e^{-iw} = -2 \Rightarrow e^{2iw} + 2e^{iw} - 1 = 0$

$e^{iw} = \dfrac{-2 \pm \sqrt{3}}{2} = (-1 + \sqrt{2})\, e^{i(0+2n\pi)}$

$iw = \ln(-1 + \sqrt{2}) + i2n\pi$

$w = \frac{1}{i} \ln(\sqrt{2} - 1) + 2n\pi$

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c) (i) $f(z) = \ln z$

$f(z) = \ln r e^{i\theta} = \ln r + i\theta$

$u = \ln r$ $\qquad v = \theta$

$u_r = \frac{1}{r}$ $\qquad v_r = 0$

$u_\theta = 0$ $\qquad v_\theta = 1$

we hape that $u_r = \frac{1}{r} v_\theta$

and $\frac{1}{r} u_\theta = -v_r$

u, v and their 1st partial derivatives exist

and continuous for all $\theta \neq 0$, $r \neq 0$

$\therefore f(z) = \ln z$ is analytic for all z, where $\theta \neq 0$ and $r \neq 0$.

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ii)

$$f(z) = \coth^{-1} z = w$$

$$\therefore z = \coth w$$

$$\frac{df}{dz} = \frac{1}{\frac{dz}{dw}} = \frac{1}{-\operatorname{cosech}^2 w}$$

$$= \cancel{\frac{1}{-\coth^2 w + 1}} = \frac{1}{1 - z^2}$$

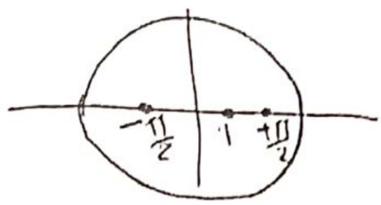
Question 3 :-

a) i) $\oint \frac{\tan z}{z-1} \cdot dz = \oint \frac{\sin z}{(z-1)\cos z}\, dz = I$

$$(z-1)\cos z = 0$$

$$\therefore z = 1 \quad , \quad z = \frac{(2n+1)}{2}\pi \quad , \quad n = 0, \pm 1, \pm 2, \ldots$$

$$I = 2\pi i \sum \operatorname{Res} = 2\pi i\,(R_1 + R_2 + R_3)$$



$$R_1 = \operatorname*{Res}_{z=1} (z-1) f(z) = \lim_{z \to 1} \tan z = \tan 1$$

$$R_2 = \operatorname*{Res}_{z=\frac{\pi}{2}} (z - \tfrac{\pi}{2}) f(z) = \lim_{z \to \frac{\pi}{2}} (z - \tfrac{\pi}{2}) \frac{\sin z}{(z-1)\cos z}$$

$$= \frac{\sin \pi/2}{(\pi/2 - 1)} \lim_{z \to \pi/2} \frac{1}{-\sin z} = \frac{-1}{(\frac{\pi}{2} - 1)}$$

$$R_3 = \operatorname*{Res}_{z=-\frac{\pi}{2}} (z + \tfrac{\pi}{2}) f(z) = \frac{1}{\pi/2 + 1}$$

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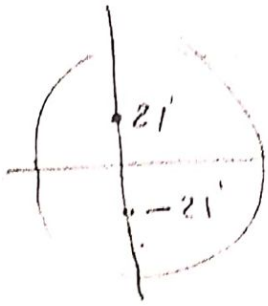
ii) $I = \oint \frac{e^{-z}}{(z^2+4)} \, dz$



$$z^2+4=0 \longrightarrow z = \pm 2i$$

$$I = 2\pi i \sum Res$$

$$= 2\pi i (R_1 + R_2)$$

$$R_1 = \underset{z=2i}{Res} = \lim_{(z \to 2i)} (z-2i) f(z)$$

$$= \lim_{z \to 2i} \left(\frac{e^{-z}}{(z+2i)} \right) = \frac{e^{-2i}}{4i}$$

$$R_2 = \underset{z=-2i}{Res} = \lim_{z \to -2i} (z+2i) f(z)$$

$$= \lim_{z \to -2i} \frac{e^{-z}}{(z-2i)} = \frac{e^{2i}}{-4i}$$

$$I = 2\pi i \left(\frac{e^{-2i}}{4i} - \frac{e^{2i}}{4i} \right) = \frac{\pi}{2} \left(e^{-2i} - e^{2i} \right)$$

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b) i) $f(z) = \ln(1+z)$ $\qquad f(0) = 0$

$f'(z) = \frac{1}{1+z}$ $\qquad f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^{-2}}$ $\qquad f''(0) = -1$

$\vdots$ $\qquad \vdots$

$$\therefore \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4!} + \cdots$$

Replacing z by $-z$, then

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$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4!} + \cdots$$

$$\ln\left(\frac{1+z}{1-z}\right) = \ln(1+z) - \ln(1-z)$$
$$= 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\right)$$

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(ii) $f(z) = (z-3)\cos\frac{1}{z+2}$

let $z+2 = t \quad \rightarrow z = t-2$

$$f(z) = (t-5)\cos\frac{1}{t}$$
$$= (t-5)\left[1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} - \cdots\right]$$
$$= t - \frac{t^3}{2!} + \frac{t^5}{4!} - \frac{t^7}{6!} - \cdots$$
$$-5 + 5\frac{t^2}{2!} - 5\frac{t^4}{4!} + 5\frac{t^6}{6!} - \cdots$$
$$= -5 + t + 5\frac{t^2}{2!} - \frac{t^3}{2!} - \frac{5}{4!}t^4 + \frac{t^5}{4!} + \cdots$$

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Question 4:-

a) The equation of a circle is:-

$$A(x^2+y^2)+Bx+Cy+D=0 \rightarrow (1)$$

$$w=\frac{1}{z} \quad , \quad \bar{w}=\frac{1}{\bar{z}}$$

$$\therefore A(z\bar{z})+B\left(\frac{z+\bar{z}}{2}\right)+C\left(\frac{z-\bar{z}}{2i}\right)+D=0$$

$$A\left(\frac{1}{w\bar{w}}\right)+B\left(\frac{w+\bar{w}}{2w\bar{w}}\right)-C\left(\frac{w-\bar{w}}{2iw\bar{w}}\right)+D=0$$

$$A+B\left(\frac{w+\bar{w}}{2}\right)-C\left(\frac{w-\bar{w}}{2i}\right)+Dw\bar{w}=0$$

$$\therefore D(u^2+v^2)+Bu-Cv+A=0 \rightarrow (2)$$

For $A=1$ and $D=0$ → equation (1) will be an equ. of a circle and from (2) it will map into a line.

For $A=0$, equation (1) will be an equation of a line and it will map into a circle from (2).

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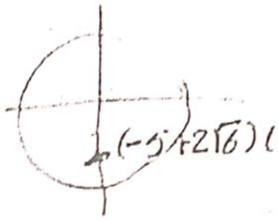
b) $I = \int_0^{2\pi} \frac{d\theta}{5+\sin\theta}$

$$\sin\theta = \frac{1}{2i}\left(Z + \frac{1}{Z}\right)$$

$$5+\sin\theta = 5 + \frac{1}{2i}\left(Z - \frac{1}{Z}\right) = \frac{1}{2iZ}\left(\cancel{2i}Z^2 + 10iZ - 1\right)$$

$$\therefore I = \oint \frac{2iZ}{Z^2+10iZ-1} \quad \frac{dZ}{iZ}$$



$$I = \oint \frac{2i\,dZ}{Z^2+10iZ-1}$$

$$Z^2+10iZ-1=0$$

$$Z = \frac{-10i \pm \sqrt{-100+4}}{2} = -5i \pm 2\sqrt{6}\,i$$

$$I = \oint \frac{2\,dZ}{(Z-a)(Z-b)}, \quad a = -5i - 2\sqrt{6}\,i, \quad b = -5i + 2\sqrt{6}\,i$$

$$= 2\pi i \operatorname*{Res}_{Z=(-5+2\sqrt{6})i} f(Z) = 2\pi i \lim_{Z\to b} \frac{2}{(Z-a)}$$

$$= 2\pi i\left(\frac{2}{-5i+2\sqrt{6}\,i+5i+2\sqrt{6}\,i}\right) = 2\pi i\left[\frac{1}{2\sqrt{6}\,i}\right]$$

$$= \frac{\pi}{\sqrt{6}} \quad \#$$

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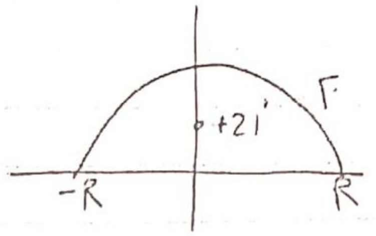
(ii) $\int_0^{\infty} \frac{1}{(x^4+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx$

let $I = \oint_C \frac{dz}{(z^2+4)^2} = 2\pi i \sum \text{Res}$



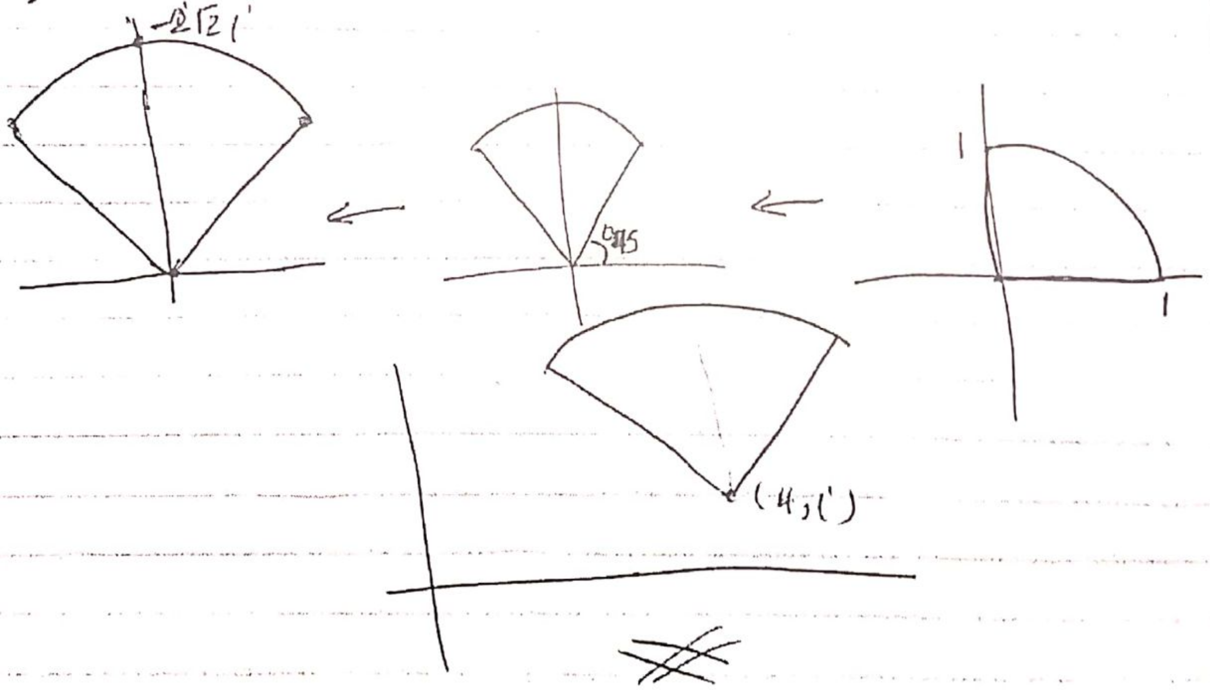
$\underset{z=2i}{\text{Res}} = \lim_{z\to 2i} \frac{d}{dz} \frac{1}{(z+2i)^2}$

$= \lim_{z\to 2i} \frac{-2}{(z+2i)^3} = \frac{-1}{32} i$

$\therefore I = 2\pi i\left(\frac{-1}{32} i\right) = \frac{\pi}{16} = \int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx + \underbrace{\int_\Gamma \frac{dz}{(z^2+4)^2}}_{0}$

$\therefore \int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{1}{2}\left(\frac{\pi}{16}\right) = \frac{\pi}{32}$

c) $w = (2+2i)z + (4+i) = 2\sqrt{2}\, e^{i\frac{\pi}{4}} z + (4+i)$



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