**The use of cubic splines in the numerical solution of fractional**

**differential equations**

W. K. Zahra[[1]](#footnote-2) and S. M. Elkholy2

1Department of Engineering Physics and Mathematics, Faculty of Engineering, Tanta Univ., Tanta, Egypt.

2Department of Engineering Physics and Mathematics, Faculty of Engineering, Kafr El Sheikh Univ., Kafr El Sheikh, Egypt.

**Abstract**:

 Fractional calculus became a vital tool in describing many phenomena appeared in physics, chemistry as well as engineering fields. Analytical solution of many applications, where the fractional differential equations appear, cannot be established. Therefore, cubic polynomial spline function based method in combined with shooting method is considered to find approximate solution for a class of fractional boundary value problems (FBPs). Convergence analysis of the method is considered. Some illustrative examples are presented.

**Keywords**: Cubic spline, Fractional boundary value problem, Dirichlet boundary conditions, Error bound.

**1. Introduction**

Fractional calculus attracted the attention of many researchers because it has recently gained popularity in the investigation of dynamical systems. There are many applications of fractional derivative and fractional integration in several complex systems such as physics, chemistry, fluid mechanics, viscoelasticity, signal processing, mathematical biology, and bioengineering and various applications in many branches of science and engineering could be found[1-16].

One of the applications where fractional differential equation appears is the equation describing the motion of fluids, which are encountered downhole during the process of oil well logging, through a device that has been designed to measure fluids viscosity. In the oil exploration industry, the fluid viscosity can indicate the permeability of the reservoir formation, its flow characteristics and the commercial value of the reservoir fluid. So, viscometers are required to measure the thermophysical properties of these fluids. It is hard to simulate



Fig.(1): MEMS instrument

reservoir conditions in a laboratory because a reservoir can exhibit temperatures of 20–200◦C and pressures of 5–200MPa. Therefore, a micro-electro-mechanical system (MEMS) instrument has been designed to measure the viscosity of fluids which contains only a single moving part (all others being electrical). This device can operate at high ambient pressures and the behavior of the device may be analyzed in a manner that allows its design to be optimised see Fig (1),[12,17].

The fluid flow is governed by the Navier–Stokes equations:

  (1.1)

Where denotes the fluid velocity,  denotes pressure,  denotes time and and are the fluid density and kinematic viscosity, respectively. Then, it was found that the equation governing the motion of the fluid through the instrument is:

,  (1.2)

The above fractional differential equation is well known as Bagley-Torvik equation when which appears in modeling the motion of a rigid plate immersed in a Newtonian fluid [12,17].

Several methods have been proposed to obtain the analytical solution of fractional differential equations (FDEs) such as Laplace and Fourier transforms, eigenvector expansion, method based on Laguerre integral formula, direct solution based on Grunewald Letnikov approximation, truncated Taylor series expansion and power series method [9,13-14,16-17,20,22]. There are also several methods have recently been proposed to solve FDEs numerically such as fractional Adams–Moulton methods, explicit Adams multistep methods, fractional difference method, decomposition method, variational iteration method, least squares finite element solution, extrapolation method and the Kansa method which is meshless, easy-to-use and has been used to handle a broad range of partial differential equation models [6-8,11-12,21,28,31].Also, the authors considered the numerical solution of the fractional boundary value problem (FBVP) , , with Dirichlet boundary conditions using quadratic polynomial spline,[34].

 The existence of at least one solution of fractional problems can be seen in [3, 11, 14-16,31].

We consider the numerical solution of the following fractional boundary value problem (FBVPs):

 ,  (1.3)

Subject to boundary conditions:

 (1.4)

where the functionis continuous on the interval  and the operator represents the Caputo fractional derivative. Where, the Caputo fractional derivative is:[17]

 . (1.5)

When  , Eq. (1.1) is reduced to the classical second order boundary value problem.

**2. Method of solution**

The following is a brief derivation of the algorithm used to solve problem (1.3-1.4). The method of solution presented in the following section is based on cubic spline approach combined with shooting method.

**2.1Cubic spline solution for FDEs**

In order to develop cubic spline approximation for the fractional differential equation (1.3)–(1.4), we would discuss the solution of (1.3) as initial value problem of the form:

,  (2.1)

 (2.2)

Let  (2.3)

be a partition of which divides the interval intoequal parts.

Cubic spline approximation will be built in each subinterval to approximate the solution of Eq.(2.1)-(2.2). Starting with the first interval , consider that the cubic polynomial spline segment  has the form:

, (2.4)

where and  are constants to be determined. It is straightforward to check:

,  and  (2.5)

By construction, Eq.(2.4) satisfies Eq.(2.1) for . Then, for complete determination of the spline in the first interval, we have to find. From Eq.(2.4), we have:

  (2.6)

We will impose that the spline be a solution of the problem (2.1) at the point. Hence, we obtain:

  (2.7)

From Eq.(2.6) , Eq.(2.7) and using (2.4) we obtain:

 (2.8)

Then the spline is fully determined in the first subinterval. In the next subinterval the cubic spline segment  has the form:

 (2.9)

From which we get:

  (2.10)

Taking into consideration that this cubic spline is of class, and again all of the coefficients of  are determined with exception of. It is easy to check that the spline be a solution of the problem (2.1) at the point, then for determining  we will impose that the spline be a solution of the problem (2.1) at the point. Hence, by repeating the previous procedure we obtain:

  (2.11)

Substituting by into Eq.(2.10) and equating the result by Eq.(2.11), we get:

(2.12)

By this way the spline is totally determined in the subinterval. Iterating this process, let us consider that the cubic spline is constructed until the subinterval  then we can define it in the next the subinterval  as:

  (2.13)

where:

 (2.14)

Then the cubic spline and easy to check that Eq.(2.13) verifies the differential equation (2.1) at the point . The constant  can be determined by imposing that the spline be a solution of the problem (2.1) at the point. Hence, we obtain:

 (2.15)

From Esq.(2.14)-(2.15), the spline approximation for the solutions of (1.1) and (2.1) at  can be written in the following form:

 (2.16)

**Lemma 1**

 Letthen the error bound associated with Eq.(2.16) is.

**Proof**

For each subinterval, the error terms are:

.

Using, Taylor expansion for, we get:

 . (2.17)

Then Eq.(2.16) and Eq.(2.17) led to:



. (2.18)

For the subinterval:

,

,

.

Then, for in Eq.(2.18), we get:

(2.19)

In general, it can be written as . Then, it can be proved that .

Cubic spline method presented above can be extended to solve fractional boundary value problems by implementing the shooting method. FBVPs (1.1) with boundary conditions (1.2) will be solved as initial value problem with two guesses for. Using linear interpolation between  in the two cases gives the next guess. Then problem (1.1) is resolved again with this new guess and so on.

**2.2 Numerical approximation of fractional term**

The algorithm used for solving fractional differential equation is based on transforming the fractional derivative into a system of ordinary differential equation. Firstly, the Caputo fractional derivative for $ f(x)$ can be written as:



We now use the binomial formula [20]:

 (2.20)

With Eq.(2.20) the expression for can be written as follows with :

 (2.21)

The integrals:

  (2.22)

are solutions to the following system of differential equations:

  (2.23)

According to Eq.(2.21-2.23) the expression for can be rewritten as:

 (2.24)

withsatisfying Eq.(2.23), Eq.(2.24) will represent the fundamental relation used in numerical representation of the fractional term in fractional differential equations. In application, we will use finite number of terms suitably chosen, so Eq.(2.24) will be:

  (2.25)

**3. Convergence analysis**

 Let be the space of cubic splines with respect toand with smoothness. Also, let us denote by the cubic spline approximation to. This implies that which can be written as.

Without loss of generality, we will consider problem (1.1) with homogeneous Dirichlet boundary conditions: [27]

 (3.1)

It will be assumed that satisfy these boundary conditions.

If we assume that the BVP along with boundary conditions (3.1) has a unique solution then there is a Green’s function for the problems

, (3.2)

where

 (3.3)

 (3.4)

where

 (3.5)

G is a compact operator, sinceis continuous in, [27].

**Lemma2**

 (3.6)

**Proof:**

From the Caputo fractional derivative , we get:





Using the principle of differentiation under the integral sign, for the function with the form 

We have that:

 

where the functions and are both continuous in both and in some region of theplane, including  and , then we can deduce that:

 

Then we have:

 

Changing the order of integration leads to:

 



 and this proves the lemma.

Substituting from (3.2-3.4) and (3.6) into (1.1), leads to:

 (3.7)

We will introduce the operator defined by:

  (3.8)

which maps  to . We also introduce a linear projection that maps  topiecewise linear interpolation at the grid points. Then Eq.(3.7) can be rewritten as:

 (3.9)

and we have also:

 (3.10)

By the definition of[27],  converges to zero as  approaches zero for continuous function. This in turn implies that  converges to zero as approaches zero.

**Theorem1** [29]

If there is large enough, then exists and consists of a sequence of bounded linear operators. Which means, for a constant independent of and , if, then.

**Theorem2** Assuming that:

(H1) The BVP (1.1) along with boundary conditions (3.1) has a unique solution in,

(H2) The BVP along with boundary conditions (3.1) has a unique solution,

then, for some we have:

 

 

Where is a constant and independent of  and and.

**Proof**

Let be a solution for Eq.(3.9) and be the solution of Eq.(1.1)-(1.2). Then, operating on both sides of Eq.(3.9) by the linear projection operator gives:

  (3.11)

Adding to both sides of Eq.(3.11) and subtracting Eq.(3.10) from the results lead to:

  (3.12)

Operating on both sides of Eq.(3.12) by leads to:

  (3.13)

Operating on both sides of Eq.(3.13) by the operator and using Eqs.(3.2-3.4) we get:

  (3.14)

Since the operator is bounded and from theorem 1 the operator is also bounded, then:

  (3.15)

From [27], we have that:

  (3.16)

  (3.17)

Where, 

Substituting from theorem1 and (3.16) into (3.15) completes the proof.

**3. Numerical examples**

We will consider some numerical examples illustrating the solution using cubic spline methods. All calculations are implemented with MATLAB 7 and we used implicit Adams-Bashforth three-step method in approximating the fractional term.

**Example 1:** Consider the initial value problem:



 . (3.1)

The analytical solution of Eq.(3.1), as found in [17], has the following form:



As, we may verify that the solution reduces to 

This example occurs in the mathematical model of (MEMS) instrument [10] and had been solved for various values of  and the solutions are represented in Figs (2-4).

Fig (2): Numerical solutions of example 1 for

Fig. 3 and Fig. 4 represent a comparison between our approximate solutions and the analytical solutions for  and  respectively. The results are tabulated also in Table 1.



Fig (3): A comparison between the analytical solution and our approximated solution for .



Fig (4): A comparison between the analytical solution and our approximated solution for .

Table 1, Numerical results of example 1

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | *k*=1 | *k*=1/5 | *k*=0.005 |
| Analytical solution | Approx solution | Analytical solution | Approx solution | Analytical solution | Approx solution |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0.125 | 0.99437 | 0.993126 | 0.992747 | 0.992391 | 0.992212 | 0.992212 |
| 0.250 | 0.979919 | 0.974802 | 0.971922 | 0.970148 | 0.968995 | 0.968983 |
| 0.375 | 0.958424 | 0.944545 | 0.938558 | 0.933609 | 0.930733 | 0.930674 |
| 0.500 | 0.930957 | 0.904813 | 0.893615 | 0.883958 | 0.878038 | 0.877899 |
| 0.625 | 0.898335 | 0.857938 | 0.838087 | 0.822499 | 0.811743 | 0.811497 |
| 0.750 | 0.861241 | 0.805442 | 0.773025 | 0.750552 | 0.732892 | 0.732514 |
| 0.875 | 0.820277 | 0.748795 | 0.699540 | 0.669584 | 0.642719 | 0.642193 |
| 1 | 0.775989 | 0.688838 | 0.618798 | 0.580978 | 0.542633 | 0.541945 |

The obtained results have good agreement with the exact solution as in Figs (3-4) and Table 1 and those published in [10].

**Example 2:** Consider the initial value problem:

 

 . (3.2)

This example had been solved for many methods. Table 2 shows a comparison between the solution of Eq.(3.2) by our method , decomposition method and fractional difference method.

 Table 2, Numerical results of example 2

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Decomp. Method[21] | Fractional diff. method[21] | Our method |
| 0 | 0 | 0 | 0 |
| 0.1 | 0.039874 | 0.039473 | 0.039933 |
| 0.2 | 0.158512 | 0.157703 | 0.158981 |
| 0.3 | 0.353625 | 0.352402 | 0.353996 |
| 0.4 | 0.622083 | 0.622083 | 0.619900 |
| 0.5 | 0.960047 | 0.957963 | 0.950455 |
| 0.6 | 1.363093 | 1.360551 | 1.348551 |
| 0.7 | 1.826257 | 1.823267 | 1.796370 |
| 0.8 | 2.344224 | 2.340749 | 2.295551 |
| 0.9 | 2.911278 | 2.907324 | 2.899808 |
| 1 | 3.521462 | 3.517013 | 3.499200 |

**Example 3:** Consider the boundary value problem:



 . (3.3)

The analytical of solution Eq.(3.3) is

 (3.4)

The numerical solutions using shooting method with  and  led to next guess of initial condition  and the results are represented in Table 3.

 Table 3, Numerical results of example 3

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Exact solution | Approx. with z3 | Error |
| 0 | 0 | 0 | 0 |
| 0.125 | 0.015625 | 0.012842 | 2.78E-3 |
| 0.250 | 0.062500 | 0.053552 | 8.95E-3 |
| 0.375 | 0.140630 | 0.138569 | 2.06E-3 |
| 0.500 | 0.250000 | 0.246192 | 3.81E-3 |
| 0.625 | 0.390630 | 0.386836 | 3.79E-3 |
| 0.750 | 0.562500 | 0.553883 | 8.62E-3 |
| 0.875 | 0.765630 | 0.758442 | 7.19E-3 |
| 1 | 1 | 0.999971 | 2.90E-5 |

**Example 4** : Consider the boundary value problem:



 . (3.5)

The analytical of solution Eq.(3.5) is (3.6)

The numerical solutions using shooting method forand with initial guessesand *,*these lead toas the fourth guess for the initial condition and the results are represented in Table 4.

 Table 4, Numerical results of example 4

|  |  |  |  |
| --- | --- | --- | --- |
| *x* | Exact solution | Approx. solution | Error |
| 0.125 | -0.00021 | -0.00039 | 1.73E-4 |
| 0.250 | -0.00292 | -0.00346 | 5.35E-4 |
| 0.375 | -0.01235 | -0.01316 | 7.98E-4 |
| 0.500 | -0.03125 | -0.03192 | 6.74E-4 |
| 0.625 | -0.05722 | -0.05713 | -9.50E-5 |
| 0.750 | -0.07910 | -0.07732 | -1.78E-3 |
| 0.875 | -0.07327 | -0.06985 | -3.42E-3 |
| 1 | 0 | 9.44E-4 | -9.44E-4 |

**Conclusion**

New scheme for solving class of fractional boundary value problem is presented using cubic spline method combined with shooting method. Transforming the fractional derivative into a system of ordinary differential equations is used for approximating the fractional term. Implicit Adams-Bashforth three-step method has been used for approximating this system of ordinary differential equations. Convergence analysis of the method is considered and is shown to be second order. Numerical comparisons between the solution using this new method and the methods introduced in [17] and [29] are presented. The obtained numerical results show that the proposed method maintain a remarkable high accuracy which make it encouraging for dealing with the solution of two-point boundary value problem of fractional order.

**Acknowledgement**

The authors are grateful to the referees for their suggestions and valuable comments.

**References**

|  |  |
| --- | --- |
| [1] | O. P. Agrawal and P. Kumar, “Comparison of five schemes for fractional differential equations,” in Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, Eds., pp. 43–60, 2007. |
| [2] | J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, The Theory of Splines and Their Applications, Academic Press, New York, NY, USA, 1967. |
| [3] | D. Baleanu and S. I. Muslih, “On Fractional Variational Principles,” in Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, Eds., pp. 115–126, 2007. |
| [4] | M. M. Benghorbal, Power series solutions of fractional differential equations and symbolic derivatives and integrals [Ph.D. thesis], Faculty of Graduate Studies, The University of Western Ontario, Ontario, Canada, 2004. |
| [5] | B. Bonilla, M. Rivero, and J. J. Trujillo, “Linear differential equations of fractional order,” in Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, Eds., pp. 77–91, 2007. |
| [6] | C. X. Jiang, J. E. Carletta, and T. T. Hartley, “Implementation of fractional-order operators on field programmable gate arrays,” in Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, Eds., pp. 333–346, 2007. |
| [7] | N. Kosmatov, “Integral equations and initial value problems for nonlinear differential,” Nonlinear Analysis: Theory, Methods & Applications, vol. 70, no. 7, pp. 2521–2529, 2009. |
| [8] | V. Lakshmikantham and A. S. Vatsala, “Basic theory of fractional differential equations,” Nonlinear Analysis:Theory, Methods & Applications, vol. 69, no. 8, pp. 2677–2682, 2008 |
| [9] | K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, JohnWiley & Sons, New York, NY, USA, 1993. |
| [10] | H. Nasuno, N. Shimizu, and M. Fukunaga, “Fractional derivative consideration on nonlinear viscoelastic statical and dynamical behavior under large pre-displacement,” in Advances in FractionalCalculus: Theoretical Developments and Applications in Physics and Engineering, J. Sabatier, O. P. Agrawal, and J. A. Tenreiro Machado, Eds., pp. 363–376, 2007. |
| [11] | A. Ouahab, “Some results for fractional boundary value problem of differential inclusions,” Nonlinear Analysis: Theory, Methods & Applications, vol. 69, no. 11, pp. 3877–3896, 2008. |
| [12] | I. Podlubny, Fractional Differential Equations, vol. 198, Academic Press, San Diego, Calif, USA, 1999. |
| [13] | I. Podlubny, I. Petr´aˇs, B. M. Vinagre, P. O’Leary, and L’. Dorˇc´ak, “Analogue realizations of fractional order controllers,” Nonlinear Dynamics, vol. 29, no. 1–4, pp. 281–296, 2002. |
| [14] | X. Su and S. Zhang, “Solutions to boundary-value problems for nonlinear differential equations of fractional order,” Electronic Journal of Differential Equations, vol. 2009, no. 26, pp. 1–15, 2009. |
| [15] | M. S. Tavazoei and M. Haeri, “A note on the stability of fractional order systems,” Mathematics and Computers in Simulation, vol. 79, no. 5, pp. 1566–1576, 2009. |
| [16] | X. Su, “Boundary value problem for a coupled system of nonlinear fractional differential equations,” Applied Mathematics Letters, vol. 22, no. 1, pp. 64–69, 2009 |
| [17] | A. D. Fitt, A. R. H. Goodwin, K. A. Ronaldson, andW. A.Wakeham, “A fractional differential equation for a MEMS viscometer used in the oil industry,” Journal of Computational and AppliedMathematics, vol. 229, no. 2, pp. 373–381, 2009 |
| [18] | J. Duan, J. An, and M. Xu, “Solution of system of fractional differential equations by Adomian decomposition method,” Applied Mathematics, A Journal of Chinese Universities Series B, vol. 22, no. 1, pp. 7–12, 2007. |
| [19] | R. Garrappa, “On some explicit Adams multistep methods for fractional differential equations,” Journal of Computational and Applied Mathematics, vol. 229, no. 2, pp. 392–399, 2009. |
| [20] | A. Ghorbani, “Toward a new analytical method for solving nonlinear fractional differential equations,” ComputerMethods in AppliedMechanics and Engineering, vol. 197, no. 49-50, pp. 4173–4179, 2008. |
| [21] | E. R. Kaufmann and E. Mboumi, “Positive solutions of a boundary value problem for a nonlinear fractional differential equation,” Electronic Journal of Qualitative Theory of Differential Equations, no. 3, pp. 1–11, 2008. |
| [22] | A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Minsk, Belarus, 1st edition, 2006. |
| [23] | S. Momani and Z. Odibat, “A novel method for nonlinear fractional partial differential equations: combination of DTM and generalized Taylor’s formula,” Journal of Computational and Applied Mathematics, vol. 220, no. 1-2, pp. 85–95, 2008. |
| [24] | W. Chen, H. Sun, X. Zhang, and D. Koroˇsak, “Anomalous diffusion modeling by fractal and fractional derivatives,” Computers & Mathematics with Applications, vol. 59, no. 5, pp. 1754–1758, 2010 |
| [25] | W. Chen, L. Ye, and H. Sun, “Fractional diffusion equations by the Kansa method,” Computers & Mathematics with Applications, vol. 59, no. 5, pp. 1614–1620, 2010. |
| [26] | K. Diethelm and G.Walz, “Numerical solution of fractional order differential equations by extrapolation,” Numerical Algorithms, vol. 16, no. 3-4, pp. 231–253, 1997. |
| [27] | G. J. Fix and J. P. Roop, “Least squares finite-element solution of a fractional order two-point boundary value problem,” Computers & Mathematics with Applications, vol. 48, no. 7-8, pp. 1017–1033, 2004 |
| [28] | L. Galeone and R. Garrappa, “Fractional Adams-Moulton methods,” Mathematics and Computers in Simulation, vol. 79, no. 4, pp. 1358–1367, 2008. |
| [29] | S. Momani and Z. Odibat, “Numerical comparison ofmethods for solving linear differential equations of fractional order,” Chaos, Solitons & Fractals, vol. 31, no. 5, pp. 1248–1255, 2007 |
| [30] | J. P. Roop, Variational solution of the fractional advection dispersion equation [Ph.D. thesis], Clemson University, Clemson, SC, USA, 2004. |
| [31] | F. I. Taukenova and M. Kh. Shkhanukov-Lafishev, “Difference methods for solving boundary value problems for fractional-order differential equations,” Computational Mathematics and Mathematical Physics, vol. 46, no. 10, pp. 1871–1795, 2006..  |
| [32] | W. K. Zahra and S. M. Elkholy, “Quadratic spline solution for boundary value problem of fractional order,” Numerical Algorithms, vol. 59, pp. 373–391, 2012. |
| [33] | P. M. Prenter, Splines and Variational Methods, JohnWiley & Sons, New York, NY, USA, 1975.  |
| [34] | R. D. Russell and L. F. Shampine, “A collocation method for boundary value problems,” Numerische Mathematik, vol. 19, pp. 1–28, 1972 |

1. Corresponding Author :waheed\_zahra@yahoo.com , wzahra@f-eng.tanta.edu.eg [↑](#footnote-ref-2)