**Quadratic spline solution for boundary value**

**problem of fractional order**

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**Abstract**

Fractional differential equations are widely applied in physics, chemistry as well as engineering fields. Therefore, approximating the solution of differential equations of fractional order is necessary. We consider the quadratic polynomial spline function based method to find approximate solution for a class of boundary value problems of fractional order. We derive a consistency relation which can be used for computing approximation to the solution for this class of boundary value problems. Convergence analysis of the method is discussed. Four numerical examples are included to illustrate the practical usefulness of the proposed method.

**Keywords:**Boundary value problems of fractional order **·**

Riemann–Liouville fractional derivative **·** Caputo fractional derivative **·**

Quadratic polynomial spline **·** Error bound

**1. Introduction**

In the last few years, considerable interest was paid to the fractional calculus, because it becomes a fundamental tool to model many dynamic systems. It was known that the classical calculus can provide a powerful tool for describing many important dynamic processes in most applied sciences, but recent studies proved that fractional calculus provides more accurate models than classical ones. There are many applications of fractional derivative and fractional integration in several complex systems such as physics, chemistry, fluid mechanics, viscoelasticity, signal processing, mathematical biology, and bioengineering and various applications in many branches of science and engineering could be found [1-5,16,19-21,24-27,31,33-34]. Boundary value problems of fractional order occur in the description of many physical processes of stochastic transport and in the investigation of liquid filtration in a strongly porous (fractal) medium [32]. Also, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamic and Cellular systems [6-7]. They occur also in the mathematical model which is developed for a micro-electro-mechanical system (MEMS) instrument that has been designed primarily to measure the viscosity of fluids that are encountered during oil well exploration [10].

Analysis and design of many systems require solution of fractional differential equations (FDEs). Several methods have recently been proposed to obtain the analytical solution of these equations. These methods include Laplace and Fourier transforms, eigenvector expansion, method based on Laguerre integral formula, direct solution based on Grunewald Letnikov approximation, truncated Taylor series expansion and power series method [9,13-14,17-18,21,23]. Also, several algorithms have been developed to solve FDEs numerically such as fractional Adams–Moulton methods, explicit Adams multistep methods, fractional difference method, decomposition method, variational iteration method , least squares finite element solution, extrapolation method and the Kansa method which is meshless, easy-to-use and has been used to handle a broad range of partial differential equation models [6-8,11-12,22,29,32]. The existence of at least one solution of fractional two-point boundary value problems can be seen in [3, 25, 31-32,34].

We consider the numerical solution of the following fractional boundary value problem (FBVP):

, , (1.1)

subject to boundary conditions:

, (1.2)

where the functionand are continuous on the interval  and the operator represents the Caputo fractional derivative. The analytical solution of (1.1-1.2) cannot be obtained for arbitrary choices of  and . When , Eq. (1.1) is reduced to the classical second order boundary value problem.

The main objective of this work is to use quadratic polynomial spline function to establish a new numerical method for the FBVP (1.1-1.2). This approach has its own advantages. For example, once the solution has been computed, the information needed for spline interpolation between mesh points is available. This is important when the solution of the boundary value problem is required at different locations in the interval. This approach has added advantage that it not only provides continuous approximations to, but also forat every point of the range of integration [2,30].

The paper is organized as follows: In section 2, we introduce some definitions and theorem necessary to our work. Derivation of our method is established in section 3. Convergence analysis of the new method is presented in section 4. In section 5, numerical results are included to show the applications and advantages of our method.

**2. Preliminaries**

In this section, definitions of fractional derivative and integral, used in our work, will be presented. There are different definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouville and the Caputo derivatives.

Let  be a function defined on, then

**Definition 1** [18]

The Riemann-Liouville fractional derivative:

.

**Definition 2** [18]

The left Riemann-Liouville fractional integral:

,

where  is the gamma function.

**Definition 3** [18,26,29]

Right Riemann-Liouville fractional integral:



**Definition 4** [4]

The Caputo fractional derivative:

.

The relation between the Riemann–Liouville operator and Caputo operator is given by:



**Theorem 1 (Leibniz' formula)**[21](p.75)

Let  be continuous on  and let  be analytical at  for all . Then, for 

,

.

Where  is the ordinary differential operator and .

**Lemma 1** [12]

If  is continuous and , then the following relationships hold:

(1) 

(2) 

(3) 



,

where, , is called incomplete gamma function.

**Lemma 2** [26,29]

If  is continuous and , then the right Riemann-Liouville fractional operators follow the following properties:

(1) ,

(2) ,

(3) 

(4) 

**Theorem 2**[19]

Let  and , and . Then for

:

* 
* 
* 
* If  with are such that, for each , there exist with,then the following composition rule holds:.

**3. Quadratic spline solution for FBVPs**

In order to develop the spline approximation for the fractional differential equation (1.1) along with the boundary condition (1.2), we, firstly, use theorem 1 [19] to convert the FBVPs given by Eq.(1.1) into the following form:

 . (3.1)

Now we introduce a finite set of grid points by dividing the interval  into equal parts.

. (3.2)

Let  be the exact solution of (2.1) and  be an approximation to  obtained by the spline functionpassing through the pointsand ().

**3.1 Spline solution for FBVPs with left fractional integral operator:**

Consider that each quadratic polynomial spline segment  has the form:

, (3.3)

where  and  are constants to be determined. To obtain the necessary conditions for the coefficients introduced in (3.3), we do not only require thatsatisfies (3.1) at  and  and that the boundary conditions (3.2) are fulfilled, but also that the value of the slope be the same for the pair of segments that join at each point  which means that satisfies the conditions:

(i) 

(ii) (3.4)

We express the three coefficients in Eq.(3.1) in terms of and [25],

   (3.5)

and we obtain:

,  and . (3.6)

Now from the continuity conditions that is the continuity of quadratic spline  at the point,, and the continuity of the first derivative, , at  and using the values of the constants in Eq.(3.6) yields to the following recurrence relation:

, (3.7)

where . (3.8)

The relation (3.7) gives () algebraic equations in the  unknown. We need two more equations, one at each end of the range of the integration interval. The two end conditions can be derived by Taylor series and the method of undetermined coefficients. These two equations are, see [28]:

. (3.9)

and

. (3.10)

**Lemma 3** [28]

Let  then the local truncation errors  associated with the scheme (3.7-3.10) are:

 (3.11)

**Proof**

To obtain the local truncation errors  of Eqns.(3.7-3.10), we first rewrite them in the form:

 (3.12)

 (3.13)

 (3.14)

The terms and, etc. in Eq.(3.12) are expanded around the point  using Taylor series and the expressions for , can be obtained. Also, expressions for , in Eq.(3.13) can be obtained by expanding these expressions around the point using Taylor series and the expressions for , can be obtained. Finally expressions  are obtained as expanding Eq.(3.14) around the point , then Eq.(3.11) is derived.

In order to determine the value of  , we use Eq.(3.8) and (3.4) and using Theorem 1 [18],we get:



Substitute the spline function for the values of ,  and  and , we get that:



 )3.15)

Then Eq.(3.8) can be written as:

. (3.16)

where,

 , ,

,and .

We need two more equations, one at each end of the range of the integration interval. The two end conditions can be derived by computing the values of the constants in Eq.(3.3) at each end with the help of Taylor series, we have that:

.

This leads to the following end conditions:

 (3.17)

Similarly, we can get the second end condition:

 , (3.18)

where , and

, .

**3.2 Spline solution for FBVPs with right fractional integral operator:**

In this case, the fractional differential equation can be written as:

 , (3.19)

along with the boundary condition (1.2).

In this case, each quadratic polynomial spline segment is:

 (3.20)

Operating on Eq.(3.19) by the operator, leads to:

 .

As indicated in section 3.1, we express the three coefficients in Eq.(3.20) in terms of and [25],

  

and we obtain:

,  and .

Applying the continuity of quadratic spline  at the point ,, and the continuity of the first derivative at the point , , yield the same recurrence relation (3.7-3.10) obtained in section 3. Where:

. (3.21)

**3.3 Spline solution for FBVPs with two sided** **fractional integral operator:**

In this case, the two sided fractional differential equation have the following form:

  (3.22)

along with the boundary condition (1.2), where .

In this case, we apply the algorithm developed in [35] to get the following system:



,

and



where , with and .

For more details about this algorithm and its convergence analysis we can refer to [35].

**4. Convergence analysis of the method**

Let and be n-dimensional column vectors. Then with these notations we can write the system given by (3.7-3.10) in matrix form as follows:

, (4.1)

Also, the system given by (3.16-3.18) can be written as follows:

, (4.2)

where the matrices , , ,  and the vector are given in Appndix1.

Rewrite Eq.(4.2) in the following form:

 (4.3)

Substitute by Eq.(4.3) into Eq.(4.1), we get the standard matrix equations as followed:

,

. (4.4)

From which the error equation can be written as:

, (4.5)

where the vector is:

 (4.6)

where *t* stands for the transpose of a matrix.

Writing the error Eq.(4.5) in the following form:



 (4.7)

For simplicity, we will consider the case where  is a constant function. Then, Eq.(3.8) becomes:

.

In order to get a bound on , we need the following lemma.

**Lemma 4** [15,28,30]

If  is square matrix of order *n* and , then exists and.

**Lemma 5**

The infinite norm of  satisfies the inequality

, (4.8)

provided that  and .

**Proof**

Rewrite the matrix given by Eq.(4.5) in the form:

,

where,

.

Then,

, (4.9)

with the help of Lemma 4, if

 (4.10)

Then

 , (4.11)

where  (4.12)

Maximum value of Eq.(4.12) at  , in this case



in order that Lemma 5 to be satisfied, the parametermust satisfy the condition:



Then

. (4.13)

The following Lemma gives a bound of .

**Lemma 6**

The matrix  given by (4.5) is nonsingular, provided that:

, (4.14)

then

. (4.15)

**Proof**

Using Lemma 5, we have:

 (4.16)

Provided that, .

It was shown that [12,25]

, (4.17)

we also have that:

 and  (4.18)

Substitute Eqns. (4.13), (4.17-4.18) into Eq. (4.16) and using Eq.(4.6), we obtain that:

 (4.19)

where, , and .

**Theorem 3**

Let be the exact solution of the continuous boundary value problem (1.1) with the boundary condition (1.2) and let , satisfy the discrete boundary value problem (4.4). Furthermore, if , then , which is given by Eq. (4.19), neglecting all errors due to round off.

**5. Numerical examples**

We now consider some numerical examples illustrating the solution using quadratic spline methods. All calculations are implemented with MATLAB 7.

**Example 1:** Consider the boundary value problem:



. (5.1)

The analytical solution of Eq.(5.1) is

 . (5.2) The numerical solutions for various values of  are represented in Fig (1) and Table (1).

Fig (1)

Table (1): Observed maximum errors for example 1 and order of convergence

(O. C.).

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *h* | α=0 | | α=0.2 | | α=0.4 | |
| Error | O. C. | Error | O. C. | Error | O. C. |
| 1/8 | 9.29E-02 |  | 1.06E-01 |  | 1.43E-01 |  |
| 1/16 | 2.57E-02 | 1.85 | 2. 91E-02 | 1.87 | 4.11E-02 | 1.80 |
| 1/32 | 7.15E-03 | 1.85 | 8.05E-03 | 1.85 | 1.10E-02 | 1.90 |
| 1/64 | 1.85E-03 | 1.95 | 2.21E-03 | 1.87 | 3.06E-03 | 1.85 |

**Example 2:** Consider the boundary value problem

  (5.3)

The analytical solution of Eq.(5.3) is:

 (5.4)

The numerical solutions for various values of  are represented in Fig (2) and Table (2).

Fig (2)

Table (2): Observed maximum errors for example 2 and order of convergence

( O. C.).

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *h* | α=0 | | α=0.2 | | α=0.4 | |
| Error | O. C. | Error | O. C. | Error | O. C. |
| 1/8 | 3.735 E-03 |  | 4.281E-03 |  | 6.875E-03 |  |
| 1/16 | 9.674 E-04 | 1.95 | 1.160E-03 | 1.88 | 1.940E-03 | 1.82 |
| 1/32 | 2.470 E-04 | 1.97 | 3.130E-04 | 1.89 | 5.310E-04 | 1.87 |
| 1/64 | 6.250E-05 | 1.98 | 8.460E-05 | 1.89 | 1.480E-04 | 1.84 |

For comparison, we plotted example 2 with various values of α and the results in [35], as shown in Figs.(3-4).



Fig (3) A comparison between the new results and the results cited in [30] for.



Fig (4) A comparison between the new results and the results cited in [30] for.

Figs (3-4) show that as increases, our new method is more accurate than methods in [35] for all values of . Moreover in the limit, as approaches zero, the scheme provides solution for the integer order system. Results also suggest that the scheme is numerically stable compared with [35].

**Example 3:** Consider the boundary value problem:



 (5.5)

The analytical solution of Eq.(5.5) is

. (5.6)

The numerical solutions for  and are represented in Fig (5), Fig (6) and Table(3).

Fig (5)

Fig(6)

Table (3): Observed maximum errors for example 3and order of convergence

( O. C.).

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *h* | α=0 | | α=0.2 | | α=0.4 | |
| Error | O. C. | Error | O. C. | Error | O. C. |
| 1/8 | 8.382E-3 |  | 8.41E-3 |  | 1.14E-2 |  |
| 1/16 | 2.155E-3 | 1.96 | 2.27E-3 | 1.89 | 3.12E-3 | 1.87 |
| 1/32 | 5.48E-4 | 1.98 | 6.11E-4 | 1.89 | 8.54E-4 | 1.87 |
| 1/64 | 1.38E-4 | 1.99 | 1.65E-4 | 1.89 | 2.35E-4 | 1.86 |

**Example 4:** Consider the boundary value problem:



 (5.7)

Where: 

The analytical solution of Eq.(5.7) is

 (5.8)

The numerical solutions for various values of  are represented in Fig (7) and Table (4).



Fig(7)

Table (4): Observed maximum errors for example 4and order of convergence

( O. C.).

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| *h* | α=0 | | α=0.2 | | α=0.4 | |
| Error | O. C. | Error | O. C. | Error | O. C. |
| 1/8 | 2.774E-2 |  | 3.175E-2 |  | 5.28E-2 |  |
| 1/16 | 7.33E-3 | 1.92 | 8.78E-3 | 1.85 | 1.47E-2 | 1.84 |
| 1/32 | 1.89E-3 | 1.96 | 2.393E-3 | 1.88 | 4.16E-3 | 1.82 |
| 1/64 | 4.81E-4 | 1.97 | 6.51E-4 | 1.89 | 1.16E-3 | 1.84 |

**Conclusion**:

New method for solving fractional boundary value problem is presented using quadratic spline method. This method is shown to be a second ordered convergent method. Convergence analysis for this method is presented. Numerical comparisons between the solution using this new method and the methods introduced in [30] are presented. The obtained numerical results show that the proposed method maintain a remarkable high accuracy which make it encouraging for dealing with the solution of two-point boundary value problem of fractional order.

**Appendix 1**

The matrices, , , K and the vector are given in Eq.(4.7) are:

, ,

, 

and



Where:

, , ,  and .

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**References**

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| [1] | Agrawal O. P. and Kumar P., Comparison of five schemes for fractional differential equations, J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and App. in Phy. and Eng. (2007) 43– 60. |
| [2 ] | Ahlberg J. H., N.Nilson E. and Walsh J. L., The theory of splines and their applications. New York: Academic Press, 1967. |
| [3] | Baleanu D. and Muslih S. I., On fractional variational principles, J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and App. in Phy. and Eng. (2007) 115 –126. |
| [4] | Benghorbal M.M., Power series solutions of fractional differential equations and symbolic derivatives and integrals, Ph.D Thesis, Faculty of Graduate studies, The University of Western Ontario, London, Ontario, 2004. |
| [5] | Bonilla B., Rivero M., and Trujillo J.J., Linear differential equations of fractional order, J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and App. in Phy. and Eng. (2007) 77-91. |
| [ 6] | Chen W., Sun H., Zhang X. and Korosak D, Anomalous diffusion modeling by fractal and fractional derivatives, Comp. Math. App., 59(2010) 1754-1758. |
| [7 ] | Chen W., Ye L. and Sun H., Fractional diffusion equation by the Kansa method, Comp. Math. App., 59(2010) 1614-1620. |
| [8] | Diethelm K., Walz G., Numerical solution of fractional order differential equations by extrapolation, Numer. Algorithms 16 (1997) 231–253. |
| [9] | Duan Junsheng D., Jianye A. and Mingyu X., Solution of system of fractional differential equations by Adomian decomposition method, App. Math. Chinese Univ. Ser. B. 22(2007) 17-12. |
| [10 ] | Fitt A.D., Goodwin A.R.H., Ronaldson K.A. and Wakeham W.A., A fractional differential equation for a MEMS viscometer used in the oil industry, J. Comput. Appl. Math, 229 (2009) 373–381. |
| [11] | Fix G.J. and Roop J.P., Least squares finite element solution of a fractional order two-point boundary value problem, Comp. Math. App., 48(2004) 1017-1033. |
| [12] | Galeone L. and Garrappa R., Fractional Adams–Moulton methods, Math. Comp. simulation 79 (2008) 1358–1367. |
| [13] | Garrappa R., On some explicit Adams multistep methods for fractional differential equations, J. Comput. Appl. Math., 229(2009)392-399 |
| [14] | Ghorbani A., Toward a new analytical method for solving nonlinear fractional differential equations, Comput. Methods Appl. Mech. Engrg., 197 (2008) 4173–4179. |
| [15] | Henrici P., Discrete variable methods in ordinary differential equations, John Wiley, NewYork, 1962. |
| [16] | Jiang C. X., Carletta J.E., and Hartley T.T., Implementation of fractional order operators on field programmable gate arrays, J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and App. in Phy. and Eng. (2007) 333–346. |
| [17] | Kaufmann E.R. AND Mboumi E. , Positive solution of a boundary value problem for a nonlinear fractional differential equations, Electronic J. of Qualitative Theory of Diff. Eq., 3 (2008) 1-11. |
| [18] | Kilbas A. A., Srivastava H. M. and Trujillo J.J., Theory of application of fractional differential equations, first ed., Belarus, 2006. |
| [19] | Kosmatov N., Integral equations and initial value problems for nonlinear differential, Nonl. Anal., Accepted 17 March 2008. |
| [20] | Lakshmikantham V. and Vatsala A.S., Basic theory of fractional differential equations, Nonl. Anal. 69 (2008) 2677–2682. |
| [21] | Miller K.S., Ross B., An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993. |
| [22] | Momani S., Odibat Z., Numerical comparison of methods for solving linear differential equations of fractional order, Chaos, Solitons and Fractals, 31(2007)1248–1255. |
| [23] | Momania S., Odibatb Z., A novel method for nonlinear fractional partial differential equations: Combination of DTM and generalized Taylor’s formula, J. Comput. Appl. Math., 220 (2008) 85 – 95. |
| [24] | Nasuno H., Shimizu N., and Fukunaga M., Fractional derivative consideration on nonlinear viscoelastic statical and dynamical behavior under large pre-displacement, J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and App. in Phy. and Eng. (2007) , 363–376. |
| [25] | Ouahab A., Some results for fractional boundary value problem of differential inclusions, Nonl. Anal., 69 (2008) 3877–3896. |
| [26] | Podlubny I., Fractional differential equation, Academic Press, San Diego, 1999. |
| [27] | Podlubny I., Petras I., Vinagre B.M., O’Leary P., Dorcak L., Analogue realizations of fractional-order controllers, Nonl. Dynam., 29 (2002) 281–296. |
| [28] | Ramadan M.A, Lashien I.F., Zahra W.K., Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems, Appl. Math. Comp., 184( 2007) 476–484,. |
| [29] | Roop J. P., Variational solution of the fractional advection dispersion equation, Ph.D Thesis, Clemson University, USA, 2004. |
| [ 30] | Sallam S. and Karaballi A.A., A quartic C3-spline collocation method for initial value problems, J. Comput. Appl. Math., 75 (1996)295-304. |
| [31] | Su X., Zhang S., Solution to boundary value problem for nonlinear differential equations of fractional order, Electron. J. Differential Equations, 26 (2009) 1–15. |
| [32] | Taukenova F. I., Shkhanukov–Lafishev M. Kh., Difference methods for solving boundary value problems for fractional differential equations, Compu. Math. Math. Phy., 46 (2006) 1785–1795. |
| [33] | Tavazoei M. S. and Haeri M., A note on the stability of fractional order systems, Math. Comp. Simulation 79 (2009) 1566-1576. |
| [34] | Xinwei S., Boundary value problem for a coupled system of nonlinear fractional, App. Math. Lett., 22 (2009) 64–69. |
| [35] | Zahra W.K., Elkholy S.M., Spline Solution to Fractional Boundary Value Problems, submitted. |

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